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The mixed two-qubit system and the structure of its ring of local invariants

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Abstract
The local invariants of a mixed two-qubit system are discussed. These invariants are polynomials in the elements of the corresponding density matrix. They are counted by means of group-theoretic branching rules which relate this problem to one arising in spin–isospin nuclear shell models. The corresponding Molien series and a refinement in the form of a four-parameter generating function are determined. A graphical approach is then used to construct explicitly a fundamental set of 21 invariants. Relations between them are found in the form of syzygies. By using these, the structure of the ring of local invariants is determined, and complete sets of primary and secondary invariants are identified: there are 10 of the former and 15 of the latter.

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1. Introduction

The invariant theory of a mixed two-qubit system has received a considerable amount of attention [1–3] in view of its importance in understanding quantum entanglement and quantum computations for such a system. Despite this, some uncertainties remain regarding the detailed nature of the ring of invariants. Here, these uncertainties are removed by providing not only a complete set of local invariants that are homogeneous polynomials in the elements of the density matrix, but also a full description of the ring of invariants expressed in terms of certain primary and secondary invariants.

Under arbitrary and independent non-singular local transformations, $g \in GL_A(2)$ and $h \in GL_B(2)$, of the two constituent qubits, the density matrix ρ of a mixed two-qubit system has matrix elements $\rho^{\beta,\lambda}_{\alpha,\kappa}$ that transform as

$$g \times h : \rho^{\beta,\lambda}_{\alpha,\kappa} \to \sum_{\gamma,\mu,\delta,\nu} g_{\alpha\gamma} h_{\kappa\lambda} \rho^{\delta,\nu}_{\gamma,\mu} g^{-1}_{\delta\beta} h^{-1}_{\nu\lambda}, \tag{1.1}$$

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with all indices varying over the two values, 1 and 2. Thus, for $G = GL_A(2) \times GL_B(2)$, the matrix elements of ρ may be regarded as a basis of a 16-dimensional G-module V on which G acts in accordance with (1.1). The local invariants of the mixed two-qubit system are then those polynomials in the elements of V which are constant under this action. These invariants and the ring $\mathcal{R}^G$ that they form are the subjects of our study.

More generally [4, 5], for any reductive linear algebraic group G and any regular finite-dimensional G-module V, Hilbert's theorem implies that the ring $\mathcal{R}^G$ of polynomials in the elements of V that are invariant under the action of G is generated by a finite set of homogeneous polynomial invariants $\{I_1, I_2, \ldots, I_N\}$. This set of invariants is said to be *fundamental* if none of the invariants $I_1, I_2, \ldots, I_N$ is redundant, in which case the invariants themselves are referred to as fundamental invariants. The ring $\mathcal{R}^G$ may be graded with respect to degree by setting $\mathcal{R}^G_m$ to be the subspace spanned by all homogeneous invariants I of degree $\deg I = m$. If this subspace has dimension $\dim \mathcal{R}^G_m = n_m$, then the corresponding Molien series is the generating function defined by

$$M(q) = \sum_{m=0}^{\infty} n_m q^m. \tag{1.2}$$

In this general setting, the ring of invariants, $\mathcal{R}^G$, necessarily has the Cohen–Macaulay property [6]. This is such that [7] in $\mathcal{R}^G$ there exists a *homogeneous system of parameters* $K_1, K_2, \ldots, K_n$, for some n, such that $\mathcal{R}^G$ is finitely generated as a free module over $\mathbb{C}[K_1, K_2, \ldots, K_n]$. It follows that in $\mathcal{R}^G$ there also exist homogeneous invariants J_k, with $k = 1, 2, \ldots, r$ for some r, such that

$$\mathcal{R}^G = \oplus_{k=0}^{r} J_k \mathbb{C}[K_1, K_2, \ldots, K_n], \tag{1.3}$$

where $J_0 = 1$. Following [7], we refer to this as a Hironaka decomposition of $\mathcal{R}^G$. The algebraically independent invariants K_i, for $i = 1, 2, \ldots, n$, are known as *primary* invariants, and the linearly independent invariants J_k, for $k = 1, 2, \ldots, r$, are known as *secondary* invariants. This indicates that an *arbitrary* invariant I can always be uniquely expressed in the form $I = \sum_{k=0}^{r} J_k P_k(K_1, K_2, \ldots, K_n)$ with each P_k a polynomial (possibly zero) in the K_i. It follows that for $|q| < 1$ the Molien series, $M(q)$, for $\mathcal{R}^G$ can be expressed as follows:

$$M(q) = \frac{\sum_{k=0}^{r} q^{\deg J_k}}{\prod_{i=1}^{n} (1 - q^{\deg K_i})}. \tag{1.4}$$

With this background, our task is to count, construct and study the polynomial invariants in the mixed two-qubit system described above. Thus, hereafter, we restrict consideration to the reductive group $G = GL_A(2) \times GL_B(2)$, and the regular G-module V spanned by the 16 components of ρ, acted upon by G as in (1.1).

Although this problem has arisen recently in the context of qubit systems, essentially the same mathematical problem has arisen previously in a quite different mathematical physics context, namely that of nuclear shell models based on the use of Wigner's spin–isospin supermultiplets [8]. In such models, multi-nucleon states are classified with respect to irreducible representations of the group $SU_{SI}(4)$ with spin–isospin subgroup $SU_S(2) \times SU_I(2)$. In this context, what is required is an integrity basis for $SU_S(2) \times SU_I(2)$ scalars in the enveloping algebra of $SU_{SI}(4)$. Such a basis has been found by Quesne [9]. In doing so, she used a method advocated by Judd *et al* [10] involving the identification of polynomial invariants of $SU_S(2) \times SU_I(2)$ constructed from the 15 generators of the Lie algebra of $SU_{SI}(4)$.

In fact, up to the inclusion of an additional invariant given by $\mathrm{Tr}(\rho)$, these invariants are in one-to-one correspondence with those of the mixed two-qubit problem. Moreover, in constructing these invariants explicitly it has been found convenient, in both the nuclear shell model and qubit contexts, to exploit the local isomorphism of the $SU(2)$ subgroup of $GL(2)$ to the corresponding rotation group $SO(3)$. This amounts to using the expansion [3]

$$\rho = t\hat{I} \otimes \hat{I} + s_a\hat{\sigma}_a \otimes \hat{I} + p_i\hat{I} \otimes \hat{\sigma}_i + b_{ai}\hat{\sigma}_a \otimes \hat{\sigma}_i, \tag{1.5}$$

where $\hat{I}$ is the 2×2 unit matrix and $\hat{\sigma}_k$ for $k = 1, 2, 3$ are the Pauli matrices and the repeated indices a, i are summed over $1, 2, 3$.

In the case of calculations relevant to the nuclear shell model, Quesne [9] not only identified a set of 20 polynomial invariants from which she speculated all others might be constructed, but also determined a three-parameter generating function for the number of linearly independent invariants separately homogeneous of given degree in the parameters s_a, p_i and b_{ai} of (1.5). Quite independently, in the qubit context, Grassl *et al* [1] identified 21 polynomial invariants, including Trρ, that they conjectured would generate the full ring of invariants $\mathcal{R}^G$. In addition, they derived a rational expression for the Molien series $M(q)$, which can be recast, although they did not do so, in the form (1.4). Then, Makhlin [3], also in the qubit context, used the expansion (1.5) to write down a set of 18 polynomial invariants that he showed were sufficient to determine the local equivalence of mixed two-qubit states. He also gave two others that together with his original 18 and Trρ were thought to generate the same polynomial ring of invariants as the 21 invariants of Grassl *et al* [1].

Given the freedom of choice in selecting a fundamental set of invariants in any ring of polynomial invariants, it is perhaps remarkable that the sets provided by Quesne [9] and Makhlin [3] coincide. Of course, under the appropriate specialization of parameters, the generating function of Quesne coincides, as it must, with the generating function obtained by Grassl *et al*. However, the Molien series, even when expressed in a form like (1.4), does not fully determine the structure of the ring of invariants [11]. To do this, it is necessary to consider the set of polynomial identities in the fundamental invariants. Such identities are known as *syzygies* [7, 11].

In this particular case, the three-parameter generating function of Quesne provides some useful additional information, especially when re-expressed, in a form attributed to Miller [12], as a sum of two terms each having only positive terms in the numerator. However, this form still does not make manifest the Cohen–Macaulay structure of the ring of polynomial invariants. In fact, the nature of this form appears to make it rather difficult to identify the required set of algebraically independent primary invariants and the corresponding set of secondary invariants.

It is these loose ends that we are at pains to tie up here. In section 2 we offer two formulae for counting the number of invariants of given degree, and then in section 3 we describe and reconcile the generating functions for the numbers of polynomial invariants given by Quesne and by Grassl *et al*. This is followed in section 4 by an account of a graphical approach [13, 14] to the identification of all possible polynomial invariants in a rather simple manner. This approach is then exploited in section 5 to prove that the invariants identified first by Quesne [9] and then by Makhlin [3] do indeed constitute a set of invariants from which all others may be polynomially generated.

That this set of invariants is a fundamental set that this is proved in section 6. This is accomplished by examining the set of syzygies relating these invariants. The syzygies themselves are, in the main, relegated to an appendix. As a consequence of this analysis, we derive an expression for the structure of the ring $\mathcal{R}^G$ that reflects the form of Quesne's three-parameter generating function. In section 7, we show that this structure can be recast in the Hironaka form, with the explicit identification of appropriate sets of primary and secondary invariants. The results are summarized in section 8.

2. Counting invariants

The 16 components, $\rho_{\alpha,\kappa}^{\beta,\lambda}$, of the mixed two-qubit density matrix form the basis of the defining irreducible representation V of $GL_{AB}(16)$. The homogeneous polynomials of degree m form the basis of the mth fold symmetrized power irreducible representation $V^{\{m\}}$ of $GL_{AB}(16)$. Let n_m be the number of linearly independent homogeneous polynomials of degree m that are invariant under the action of the local transformations of $GL_A(2) \times GL_B(2)$. Then, n_m is just the number of times the trivial, one-dimensional irreducible representation, $V^{\{0\}\cdot\{0\}}$, of $GL_A(2) \times GL_B(2)$ appears in the restriction of $V^{\{m\}}$ from $GL_{AB}(16)$ to this subgroup.

Quite generally [4, 15], for all n, the polynomial irreducible representations of $GL(n)$ have characters $\{\lambda\}$, specified by partitions λ of length $\ell(\lambda) \leqslant n$, which are nothing other than Schur functions $s_\lambda(x_1, \ldots, x_n)$ of the eigenvalues x_i of the group elements of $GL(n)$. In addition, there exist corresponding contragredient irreducible representations with characters $\{\overline{\lambda}\}$, which are Schur functions $s_\lambda(\overline{x}_1, \ldots, \overline{x}_n)$ of the eigenvalues $\overline{x}_i = x_i^{-1}$ of the inverse group elements.

We consider the following group–subgroup chains:

$$\begin{array}{ccc} & GL_{AB}(16) & \\ \swarrow & & \searrow \\ GL_A(4) \times GL_B(4) & & GL_{AB}(4) \times \overline{GL_{AB}(4)} \\ \downarrow & & \downarrow \\ GL_A(2) \times \overline{GL_A(2)} \times GL_B(2) \times \overline{GL_B(2)} & & GL_A(2) \times GL_B(2) \times \overline{GL_A(2)} \times \overline{GL_B(2)} \\ \searrow & & \swarrow \\ & GL_A(2) \times GL_B(2), & \end{array} \tag{2.1}$$

where each $\overline{GL(n)}$ is isomorphic to $GL(n)$, but the overline is used to indicate that in accordance with (1.1) the action of $\overline{GL(n)}$ is contragredient to that of $GL(n)$.

The corresponding branching rules for the left- and right-hand chains take the form [16]

$$\begin{aligned} \{m\} &\to \sum_\lambda \{\lambda\} \cdot \{\lambda\} \to \sum_{\lambda,\mu,\nu,\sigma,\tau} \left(k_{\mu\nu}^{\lambda}\{\mu\} \cdot \{\overline{\nu}\}\right) \cdot \left(k_{\sigma\tau}^{\lambda}\{\sigma\} \cdot \{\overline{\tau}\}\right) \\ &\to \sum_{\lambda,\mu,\nu,\sigma,\tau} k_{\mu\nu}^{\lambda} k_{\sigma\tau}^{\lambda} (\cdots + \delta_{\mu\nu}\{0\}) \cdot (\cdots + \delta_{\sigma\tau}\{0\}) \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} \{m\} &\to \sum_\lambda \{\lambda\} \cdot \{\overline{\lambda}\} \to \sum_{\lambda,\mu,\nu,\sigma,\tau} \left(k_{\mu\nu}^{\lambda}\{\mu\} \cdot \{\nu\}\right) \cdot \left(k_{\sigma\tau}^{\lambda}\{\overline{\sigma}\} \cdot \{\overline{\tau}\}\right) \\ &\to \sum_{\lambda,\mu,\nu,\sigma,\tau} k_{\mu\nu}^{\lambda} k_{\sigma\tau}^{\lambda} (\cdots + \delta_{\mu\sigma}\{0\}) \cdot (\cdots + \delta_{\nu\tau}\{0\}). \end{aligned} \tag{2.3}$$

In each case, the coefficients $k_{\mu\nu}^{\lambda}$ are defined by the inner product rule $\chi_\kappa^\mu \chi_\kappa^\nu = \sum_\lambda k_{\mu\nu}^{\lambda} \chi_\kappa^\lambda$ for characters χ_κ^λ of irreducible representations of the symmetric group S_m. The identical nature of the coefficients in each branching is a consequence of the fact that they are determined by the symmetry types ρ of the relevant characters $\{\rho\}$ or $\{\overline{\rho}\}$, with no distinction between a representation and its contragredient. In these expressions, $\lambda, \mu, \nu, \sigma, \tau$ are partitions of m, restricted by the conditions $\ell(\lambda) \leqslant 4$ and $\ell(\mu), \ell(\nu), \ell(\sigma), \ell(\tau) \leqslant 2$. Picking out the coefficient of $\{0\} \cdot \{0\}$ in these branching rules, (2.2) and (2.3), gives rise to the following formulae for the required multiplicities:

$$n_m = \sum_{\lambda \vdash m;\ell(\lambda)\leqslant 4} \left(\sum_{\mu \vdash m;\ell(\mu)\leqslant 2} k^{\lambda}_{\mu\mu} \right)^2 = \sum_{\lambda \vdash m;\ell(\lambda)\leqslant 4} \left(\sum_{\mu,\nu \vdash m;\ell(\mu),\ell(\nu)\leqslant 2} \left(k^{\lambda}_{\mu\nu}\right)^2 \right). \tag{2.4}$$

Stimulated by the work of Davis *et al* [2], Wybourne used the first of these formulae to calculate the specific values of n_m using his software package SCHUR [17], and thereby built up the corresponding Molien series term by term [18]

$$\begin{aligned} M(q) = 1 + q + 4q^2 + 6q^3 + 16q^4 + 23q^5 + 52q^6 + 77q^7 + 150q^8 + 224q^9 + 396q^{10} + 583q^{11} \\ + 964q^{12} + 1395q^{13} + 2180q^{14} + 3100q^{15} + 4639q^{16} + 6466q^{17} + 9344q^{18} \\ + 12\,785q^{19} + 17\,936q^{20} + 24\,121q^{21} + 33\,008q^{22} + 43\,674q^{23} + 58\,512q^{24} \\ + 76\,277q^{25} + 100\,312q^{26} + 129\,009q^{27} + 166\,932q^{28} + 212\,022q^{29} \\ + 270\,448q^{30} + O(q^{31}). \end{aligned} \tag{2.5}$$

3. Generating functions

To go further, it is preferable to derive the corresponding generating function. To this end, we can use Molien's theorem [7, 11, 19]. This states that if G is a compact continuous group, with elements g and Haar measure $\mathrm{d}\mu(g)$, then ([7], p 188)

$$M(q) = \int_{g\in G} \frac{\mathrm{d}\mu(g)}{\det(I - qg)}. \tag{3.1}$$

In our case, G is, in principle, the non-compact group $GL_A(2) \times GL_B(2)$. However, the number of invariants is unchanged if we restrict from $GL_A(2) \times GL_B(2)$ to the compact subgroup $SU_A(2) \times SU_B(2)$. This is so firstly, because the action (1.1) allows us to restrict, without loss of generality, to the unimodular elements of $SL_A(2) \times SL_B(2)$. Then secondly, because all the irreducible and inequivalent representations of the non-compact group $SL(2)$ that we encounter are components of tensor powers of the defining representation, and thus remain irreducible and inequivalent on restriction to the compact subgroup $SU(2)$. Moreover, the invariants themselves only arise from tensor products of pairs of irreducible representations and their contragredients of the form $\{\mu\} \cdot \{\overline{\mu}\}$ and $\{\sigma\} \cdot \{\overline{\sigma}\}$. All the irreducible representations that appear in these products are, in fact, faithful irreducible representations not of $SU_A(2) \times SU_B(2)$ but of $SO_A(3) \times SO_B(3)$. This allows us to take $G = SO_A(3) \times SO_B(3)$, with the group element g acting on the 16-dimensional density matrix realized as the tensor product of two 4×4 matrices. This group element can be diagonalized to yield eigenvalues $\{1, \mathrm{e}^{\mathrm{i}\theta}, 1, \mathrm{e}^{-\mathrm{i}\theta}\} \times \{1, \mathrm{e}^{\mathrm{i}\phi}, 1, \mathrm{e}^{-\mathrm{i}\phi}\}$. The corresponding measure can be written in the form $\mathrm{d}\mu(g) = (1/4\pi^2)(1 - \cos\theta)(1 - \cos\phi)\,\mathrm{d}\theta\,\mathrm{d}\phi$ [15]. Setting $z = \mathrm{e}^{\mathrm{i}\theta}$ and $w = \mathrm{e}^{\mathrm{i}\phi}$ then leads to integrations around $|z| = 1$ and $|w| = 1$. Molien's theorem then gives

$$\begin{aligned} M(q) = \frac{1}{(4\pi \mathrm{i})^2} \oint_{|z|=1} \oint_{|w|=1} \frac{(1-z)^2(1-w)^2 z^{-2} w^{-2}}{(1-q)^4(1-qz)^2(1-q/z)^2(1-qw)^2(1-q/w)^2} \\ \times \frac{\mathrm{d}w\,\mathrm{d}z}{(1-qzw)(1-qz/w)(1-qw/z)(1-q/zw)}. \end{aligned} \tag{3.2}$$

For $|q| < 1$, the repeated use of Cauchy's residue theorem then yields the result

$$M(q) = \frac{1 - q^2 - q^3 + 2q^4 + 2q^5 + 2q^6 - q^7 - q^8 + q^{10}}{(1-q)^{10}(1+q)^6(1+q^2)^2(1+q+q^2)^3} \tag{3.3}$$

$$= \frac{1 + q^4 + q^5 + 3q^6 + 2q^7 + 2q^8 + 3q^9 + q^{10} + q^{11} + q^{15}}{(1-q)(1-q^2)^3(1-q^3)^2(1-q^4)^3(1-q^6)}, \tag{3.4}$$

where the first form is due to Grassl *et al* [1], while the second form, which we introduced elsewhere [14], emphasizes the fact that, for $|q| < 1$, the expansion as a power series in q yields a series in which all the coefficients n_m of q^m are non-negative integers, as required.

This generating function may be refined in the manner prescribed by Quesne [9]. To this end, one merely weights the contributions associated with the four terms in expansion (1.5) of ρ by factors a, s, p and b, rather than just by q. This gives the generating function

$$G(a,s,p,b) = \frac{1}{(4\pi \mathrm{i})^2}\oint_{|z|=1}\oint_{|w|=1} \frac{(1-z)^2(1-w)^2 z^{-2} w^{-2}\,\mathrm{d}w\,\mathrm{d}z}{\begin{array}{l}((1-a)(1-sz)(1-s)(1-s/z)\\ (1-pw)(1-bwz)(1-bw)(1-bw/z)\\ (1-p)(1-bz)(1-b)(1-b/z)\\ (1-p/w)(1-bz/w)(1-b/w)(1-b/wz))\end{array}}. \tag{3.5}$$

This time the repeated use of Cauchy's residue theorem yields

$$G(a,s,p,b) = \frac{\begin{array}{c}1 + sp(b^2+b^3) + (s^2p + sp^2)(b^3+b^4+b^5)\\ +(s^3+p^3)b^6 + s^2p^2b^5 - s^2p^2b^8 - (s^4p + sp^4)b^7\\ -(s^3p^2+s^2p^3)(b^8+b^9+b^{10}) - s^3p^3(b^{10}+b^{11}) - s^4p^4b^{13}\end{array}}{\begin{array}{c}((1-a)(1-s^2)(1-p^2)(1-b^2)(1-spb)(1-b^3)(1-s^2b^2)(1-p^2b^2)(1-b^4)\\ (1-s^2b^4)(1-p^2b^4))\end{array}} \tag{3.6}$$

$$= \frac{\begin{array}{c}(1 + spb^2 + spb^3 + s^2p^2b^5)[1 + s^2b^4/(1-s^2b^4) + p^2b^4/(1-p^2b^4)]\\ + (sp^2b^3 + s^2pb^4 + s^2pb^5 + s^3b^6)/(1-s^2b^4)\\ + (s^2pb^3 + sp^2b^4 + p^2sb^5 + p^3b^6)/(1-p^2b^4)\end{array}}{(1-a)(1-s^2)(1-p^2)(1-b^2)(1-spb)(1-b^3)(1-s^2b^2)(1-p^2b^2)(1-b^4)}. \tag{3.7}$$

Apart from the factor $1/(1-a)$ associated with the invariant $\mathrm{Tr}\,\rho$, the first form is due to Quesne [9] and the second, while derived independently here, is a modest, but useful rearrangement of the terms in the expression due to Miller that is quoted by Gaskell *et al* [12]. Once again this second form (3.7) emphasizes the fact that the expansion of $G(a,s,p,b)$ for $|a|$, $|s|$, $|p|$ and $|b|$ all < 1 yields a series in which all the coefficients n_{m_a,m_s,m_p,m_b} of $a^{m_a}s^{m_s}p^{m_p}b^{m_b}$ are non-negative integers. Of course, on setting $a = s = p = b = q$, one recovers the Molien series (3.4).

4. Construction of the invariants

We now take advantage of expansion (1.5) of ρ in terms of Pauli matrices to construct $SO_A(3) \times SO_B(3)$ invariants as polynomials in the 16 components of the parameters

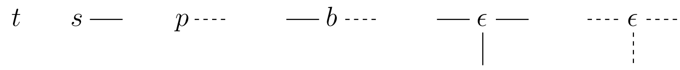

Figure 1. Graphical representation of t, s_a, p_i, b_{ai}, ϵ_{abc} and ϵ_{ijk}.

t, $\mathbf{s} = (s_1, s_2, s_3)$, $\mathbf{p} = (p_1, p_2, p_3)$ and $\mathbf{b} = (b_{ai})_{1\leqslant a,i\leqslant 3}$. In doing this, use may be made of the two distinct Levi-Civita antisymmetric tensors ϵ_{abc} and ϵ_{ijk} associated with $SO_A(3)$ and $SO_B(3)$, respectively. The property characterizing an invariant is that following contraction there must remain no free indices of any kind.

In enumerating and studying the properties of invariants, it is convenient to adopt a graphical approach to the coupling of tensors [20] that has been introduced elsewhere [14] in the present context. The building blocks to be used in the construction of invariants are given in graphical form in figure 1. Algebraically, these building blocks correspond to: the scalar t; the vectors s_a and p_i; the matrices b_{ai} and b^{T}_{ia}, as appropriate; and antisymmetric tensors ϵ_{abc} and ϵ_{ijk}.

The graphs of the invariants are then obtained by connecting together all the edges of the building blocks to create graphs with no free edges. Each edge corresponds to a contraction of the appropriate common index associated with the two vertices that it links. We use the convention that solid lines represent contractions over indices of type $a, b, \ldots$, associated with s_a and ϵ_{abc}, and dotted lines represent contractions over indices of type $i, j, \ldots$, associated with p_i and ϵ_{ijk}. Guided in part by the form of the generating function $G(a, s, p, b)$ in which each term $a^{m_a}s^{m_s}p^{m_p}b^{m_b}$, whether appearing in the numerator or denominator, is to be associated with an invariant that is of degree m_a, m_s, m_p and m_b in the components of t, $\mathbf{s}$, $\mathbf{p}$ and $\mathbf{b}$, respectively, one rapidly arrives at the candidate set of fundamental invariants represented graphically in figure 2.

Of course, many other invariants may be constructed by means of this graphical approach. However, as we show in section 5, all these others may be expressed as polynomials in those of figure 2. Algebraically, the invariants of figure 2 take the form given in table 1, where the correspondence with the invariants of Quesne [9] and Makhlin [3] is given. In specifying the invariants, we have adopted a repeated index convention to signify contractions over both types of indices $a, b, \ldots$ and $i, j, \ldots$, that is to say, to signify in each case sums over the index values 1, 2 and 3.

Of these invariants, $K_1, \ldots, K_9$ together with X_1 and X_2 are of precisely the right degrees to correspond to the 11 factors appearing in the denominator of $G(a, s, p, b)$, as given by (3.6), while for each of the remaining 10 invariants U_i, V_j and W_k for $i = 1, 2$ and $j, k = 1, 2, 3, 4$, there exists a positive term of the corresponding degree in the numerator. However, there also exists in the numerator a further positive term, as well as many negative terms. As pointed out by Quesne [9], while the additional positive term can be associated with the product U_1U_2, certain relations between powers and products of the above invariants must be responsible for the negative terms. Having given one such relation, in the form of a syzygy involving U_1^2, she said of the other relations that ‘owing to their high degree of complexity, we have not explored them further’. It is this that we shall do in section 7, but first it is necessary to prove that the set of invariants given in table 1 is complete in the sense that all others may be expressed polynomially in terms of these.

5. Completeness of the candidate set of invariants

It is convenient to refer to the vectors which transform under the two different groups, $SO_A(3)$ and $SO_B(3)$, as generic S-vectors and P-vectors, respectively, with S-type indices denoted

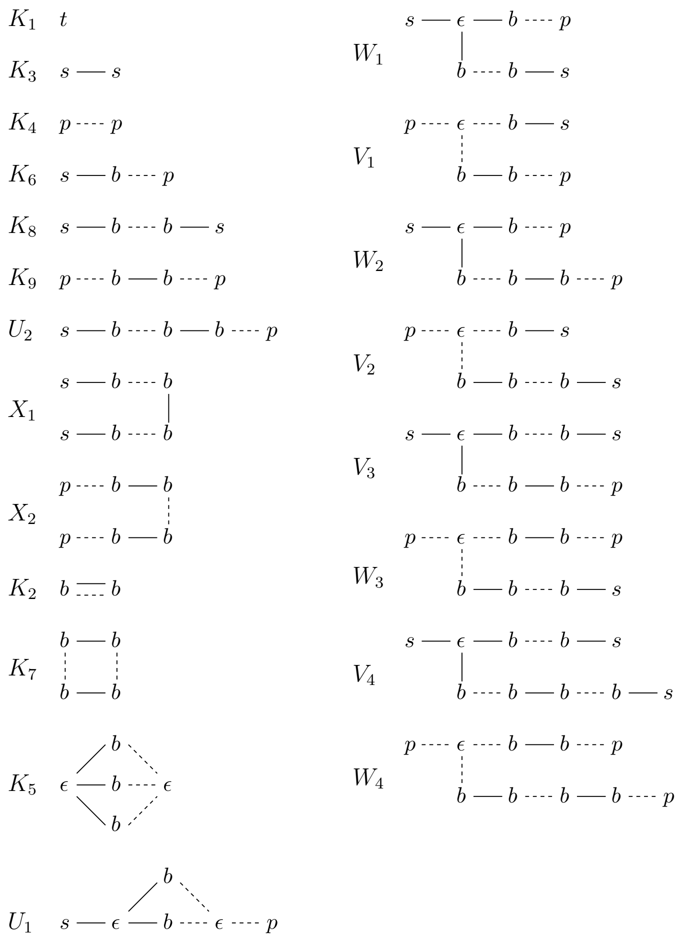

Figure 2. Graphical representation of invariants.

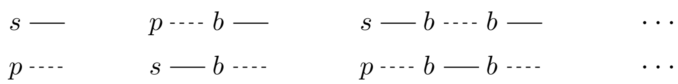

Figure 3. *S*-vectors and *P*-vectors constructed from s, p and b.

by $a, b, c, \ldots$ and *P*-type indices by $i, j, k, \ldots$. In this way, vectors of *S*-type, with one free *S*-type index, include those of the form s_a, $p_i b^{\mathrm{T}}_{ia}$, $s_b b_{bi} b^{\mathrm{T}}_{ia}, \ldots$, with s and p being contracted with products of even and odd numbers of matrices b or b^{T}, respectively. Vectors of *P*-type, with one free *P*-type index include those of the form p_i, $s_a b_{ai}$, $p_j b_{ja} b^{\mathrm{T}}_{ai}, \ldots$, with s and p now being contracted with products of odd and even numbers of matrices b or b^{T}, respectively. These vectors then have the graphical representation displayed in figure 3.

A fundamental set of polynomial invariants is any set, with no redundancies, from which all others may be obtained as sums of products and powers. Disconnected graphs are thus excluded since the corresponding invariant is just the product of the invariants specified by their connected components. Thus, the only fundamental invariant involving the scalar t is t itself. In other cases, the presence of certain subgraphs may be excluded by virtue of the

Table 1. Candidate fundamental invariants.

Label	Quesne	Makhlin	Invariant
K_1			t
K_2	$C^{(002)}$	I_2	$b_{ai}b^{\mathrm{T}}_{ia}$
K_3	$C^{(200)}$	I_4	$s_a s_a$
K_4	$C^{(020)}$	I_7	$p_i p_i$
K_5	$C^{(003)}$	$6I_1$	$\epsilon_{abc}\epsilon_{ijk}b_{ai}b_{bj}b_{ck} = 6\det b$
K_6	$C^{(111)}$	I_{12}	$s_a b_{ai} p_i$
K_7	$C^{(004)}$	I_3	$b_{ai}b^{\mathrm{T}}_{ib}b_{bj}b^{\mathrm{T}}_{ja}$
K_8	$C^{(202)}$	I_5	$s_a b_{ai}b^{\mathrm{T}}_{ib}s_b$
K_9	$C^{(022)}$	I_8	$p_i b^{\mathrm{T}}_{ia}b_{aj}p_j$
X_1	$C^{(204)}$	I_6	$s_a b_{ai}b^{\mathrm{T}}_{ib}b_{bj}b^{\mathrm{T}}_{jc}s_c$
X_2	$C^{(024)}$	I_9	$p_i b^{\mathrm{T}}_{ia}b_{aj}b^{\mathrm{T}}_{jb}b_{bk}p_k$
U_1	$C^{(112)}$	I_{14}	$\epsilon_{abc}\epsilon_{ijk}s_a b_{bj}b_{ck}p_i$
U_2	$C^{(113)}$	I_{13}	$s_a b_{ai}b^{\mathrm{T}}_{ib}b_{bj}p_j$
V_1	$C^{(123)}$	I_{16}	$\epsilon_{ijk}p_i b^{\mathrm{T}}_{ja}s_a b^{\mathrm{T}}_{kb}b_{bl}p_l$
V_2	$C^{(214)}$	I_{17}	$\epsilon_{ijk}p_i b^{\mathrm{T}}_{ja}s_a b^{\mathrm{T}}_{kb}b_{bl}b^{\mathrm{T}}_{lc}s_c$
V_3	$C^{(215)}$	I_{19}	$\epsilon_{abc}s_a b_{bi}b^{\mathrm{T}}_{id}s_d b_{cj}b^{\mathrm{T}}_{je}b_{ek}p_k$
V_4	$C^{(306)}$	I_{10}	$\epsilon_{abc}s_a b_{bi}b^{\mathrm{T}}_{id}s_d b_{cj}b^{\mathrm{T}}_{je}b_{ek}b^{\mathrm{T}}_{kf}s_f$
W_1	$C^{(213)}$	I_{15}	$\epsilon_{abc}s_a b_{bi}p_i b_{cj}b^{\mathrm{T}}_{jd}s_d$
W_2	$C^{(124)}$	I_{18}	$\epsilon_{abc}s_a b_{bj}p_j b_{ck}b^{\mathrm{T}}_{kd}b_{dl}p_l$
W_3	$C^{(125)}$	I_{20}	$\epsilon_{ijk}p_i b^{\mathrm{T}}_{ja}b_{al}p_l b^{\mathrm{T}}_{kb}b_{bm}b^{\mathrm{T}}_{mc}s_c$
W_4	$C^{(036)}$	I_{11}	$\epsilon_{ijk}p_i b^{\mathrm{T}}_{ja}b_{al}p_l b^{\mathrm{T}}_{kb}b_{bm}b^{\mathrm{T}}_{mc}b_{cn}p_n$

corresponding invariant vanishing identically, or being expressible as a linear sum of products and powers. In particular, the following facts should be noted.

- The antisymmetry of the two ϵ-tensors implies that $\epsilon_{abc}S_aS_b = 0$ and $\epsilon_{ijk}P_iP_j = 0$, for any vectors S and P having just a single uncontracted S- and P-type index, respectively. This ensures that no two edges of an ϵ node may be attached to identical vectors.
- This antisymmetry also guarantees that $\epsilon_{abc}b_{ai}b^{\mathrm{T}}_{id}\cdots b_{ej}b^{\mathrm{T}}_{jb}$ and $\epsilon_{ijk}b^{\mathrm{T}}_{ia}b_{al}\cdots b^{\mathrm{T}}_{mb}b_{bj}$ are both identically zero, since these products necessarily involve an even number of b's or b^{T}'s, and such a product is always a symmetric matrix. It follows that no two edges of an ϵ node may be linked to a loop of b nodes.
- The three edges of any ϵ node cannot all be linked to b nodes, except for the case $K_5 = 6\det(b)$ illustrated in figure 2, because in every other case a factor of $\det(b)$ can be extracted.
- All graphs containing two ϵ nodes of the same type may be eliminated by means of the identities

$$\epsilon_{abc}\epsilon_{def} = \delta_{ad}\delta_{be}\delta_{cf} + \delta_{ae}\delta_{bf}\delta_{cd} + \delta_{af}\delta_{bd}\delta_{ce} - \delta_{ad}\delta_{bf}\delta_{ce} - \delta_{ae}\delta_{bd}\delta_{cf} - \delta_{af}\delta_{be}\delta_{cd}; \quad (5.1)$$

$$\epsilon_{ijk}\epsilon_{lmn} = \delta_{il}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{kl} + \delta_{in}\delta_{jl}\delta_{km} - \delta_{il}\delta_{jn}\delta_{km} - \delta_{im}\delta_{jl}\delta_{kn} - \delta_{in}\delta_{jm}\delta_{kl}. \quad (5.2)$$

There exist three particular generic identities illustrated graphically in figures 4 and 5.

In each of these identities, S and P are arbitrary S- and P-type vectors as exemplified above in figure 3, but including more generally the case of any vectors constructed out of s, p, b and the two ϵ's, having just one free index of type S or P, respectively. In fact, all of these identities also remain valid if each factor S_aP_i is replaced by any B_{ai}, where B is a completely arbitrary matrix, including any matrix constructed as a product of any odd number of b's and b^{T}'s.

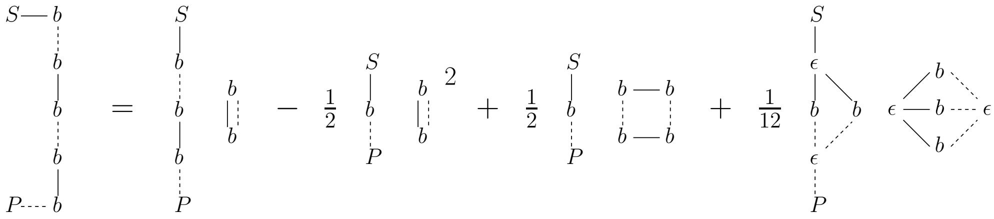

Figure 4. Elimination of strings of five b nodes.

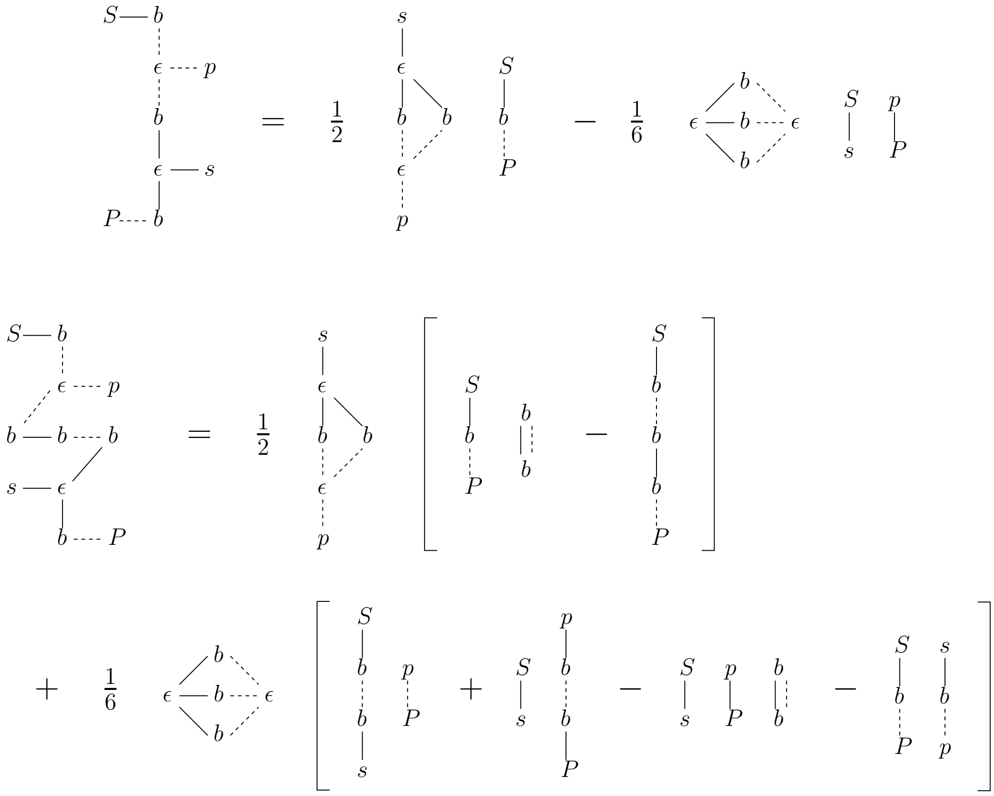

Figure 5. Simplification of invariants involving both ϵ nodes.

- The identity of figure 4 implies that the graph of any fundamental invariant may contain a string of no more than four b nodes. This is a stronger statement than the fact that the Cayley–Hamilton theorem disallows strings of six b nodes by expressing the cube of the 3×3 matrix bb^{T} in the form

$$(bb^{\mathrm{T}})^3 = K_2(bb^{\mathrm{T}})^2 + \tfrac{1}{2}\big(K_2^2 - K_7\big)(bb^{\mathrm{T}}) + \tfrac{1}{36}K_5^2 I, \qquad (5.3)$$

  where I is the 3×3 unit matrix.

- The two identities of figure 5 imply that there can be no more than two fundamental invariants involving both ϵ's, namely those corresponding to $K_5 = \epsilon_{abc}\epsilon_{ijk}b_{ai}b_{bj}b_{ck}$ and $U_1 = \epsilon_{abc}\epsilon_{ijk}s_a p_i b_{bj} b_{ck}$.

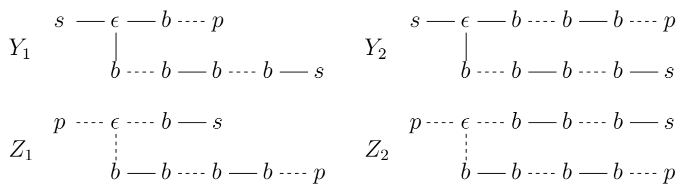

Figure 6. Graphs of candidate fundamental invariants, later excluded.

With the above restrictions, the only possible fundamental invariants involving just a single ϵ are those in which the three edges attached to the ϵ involve strings containing different numbers of b nodes, one of which is zero and the others lie between 1 and 4.

It follows that the graphs corresponding to candidate fundamental invariants are just those displayed in figure 2 together with those of figure 6.

However, explicit calculations reveal the following identities:

$$\begin{aligned} Y_1 &= V_3 + K_2 W_1, \qquad & Y_2 &= \tfrac{1}{6} K_5 V_2 + \tfrac{1}{2} W_1 \big(K_2^2 - K_7\big), \\ Z_1 &= W_3 + K_2 V_1, \qquad & Z_2 &= \tfrac{1}{6} K_5 W_2 + \tfrac{1}{2} V_1 \big(K_2^2 - K_7\big). \end{aligned} \tag{5.4}$$

These enable each of the invariants of figure 6 to be eliminated in favour of those of figure 2, which thus illustrates a set of 21 invariants from which all others may be constructed as sums of products and powers. This proves the validity of the conjecture of Grassl *et al* [1] that a set of 21 invariants suffices to generate all invariants. In fact, we have

Theorem 5.1. *Let V be the vector space spanned by the 16 elements of the two-qubit density matrix ρ. Then, the ring $\mathcal{R}^G$ of polynomials in the elements of V that are invariant under the local action of $G = GL_A(2) \times GL_B(2)$ is generated by the 21 invariants of table 1:*

$$\{K_1, \ldots, K_9\} \cup \{X_1, X_2\} \cup \{U_1, U_2\} \cup \{V_1, \ldots, V_4\} \cup \{W_1, \ldots, W_4\}. \tag{5.5}$$

6. Syzygies and fundamental invariants

It should be noted that although the ring of invariants, $\mathcal{R}^G$, is generated by the invariants of table 1, it is not freely generated by them. Indeed, $\mathcal{R}^G$ is isomorphic to $\mathcal{P}/\ker\phi$, where $\mathcal{P}$ is the polynomial ring over algebraically independent invariants that are in one-to-one correspondence with the terms $K_1, K_2, \ldots, W_4$ enumerated graphically in figure 2 and ϕ is the map $\phi : \mathcal{P} \to \mathcal{R}^G$ defined by evaluating each of the invariants as polynomials in the parameters t, s_a, p_i and b_{ia} as given in table 1. In order to identify $\ker\phi$, it is necessary to consider polynomial identities in the invariants when evaluated under the action of ϕ. These identities are known as *syzygies*. More properly, these syzygies are known as the syzygies of the first kind [11]. Further polynomial relationships between the syzygies of the first kind are known as the syzygies of the second kind, and so on. It is only the syzygies of the first kind that are required here.

A computer search has identified 63 syzygies of the first kind. As we show below, they enable the structure of $\mathcal{R}^G$ to be fully determined. These syzygies take the form $\phi(SY[k]) = 0$ for $k = 1, 2, \ldots, 63$, where each $SY[k]$ is an independent polynomial in the 21 invariants of table 1. The $SY[k]$ are listed in the appendix. It may be observed that each $SY[k]$ is homogeneous in the components of t, $\mathbf{s}$, $\mathbf{p}$ and $\mathbf{b}$, in the sense that each of its summands is homogeneous and of the same multidegree (m_a, m_s, m_p, m_b) in these variables. Particular instances are

$$SY[1] := U_1^2 + 8U_2K_6 - 4K_2K_6^2 - 4K_8K_9 + 4K_2K_3K_9 + 4K_2K_4K_8 + 2K_3K_4K_7 - 2K_2^2K_3K_4 - 4X_1K_4 - 4X_2K_3; \qquad (6.1)$$

$$SY[2] := U_2^2 - 2X_1K_9 - 2X_2K_8 + 2K_2K_8K_9 + K_6^2K_7 - K_2^2K_6^2 - 2K_3K_4K_5^2 + 2U_1K_5K_6; \qquad (6.2)$$

$$SY[43] := 2X_1X_2 - 2X_1K_2K_9 - 2X_2K_2K_8 + K_7K_8K_9 + K_2^2K_8K_9 - 2U_2K_6K_7 + 2U_2K_2^2K_6 + K_2K_6^2K_7 - K_2^3K_6^2 - 2U_1U_2K_5 + 2U_1K_2K_5K_6 - 2K_5^2K_6^2 + 2K_4K_5^2K_8 + 2K_3K_5^2K_9 - 2K_2K_3K_4K_5^2. \qquad (6.3)$$

The first of these was found by Quesne [9], but the other two and the remaining 60 are new. As indicated in the appendix, these syzygies allow each of the following to be eliminated in favour of other terms of $\mathcal{R}^G$:

$$U_i^2, \quad U_iV_j, \quad U_iW_k, \quad V_jV_k, \quad V_jW_k, \quad W_jW_k, \quad V_jX_2, \quad W_kX_1, \quad X_1X_2, \qquad (6.4)$$

for $1 \leqslant i \leqslant 2$ and $1 \leqslant j, k \leqslant 4$, but neither U_1U_2 nor any of the following:

$$V_jX_1^m, \qquad W_kX_2^m, \qquad X_1^m, \qquad X_2^m, \qquad (6.5)$$

for $1 \leqslant j, k \leqslant 4$ and any $m \geqslant 1$.

To make use of the syzygies, let

$$\mathcal{S} = \langle SY[1], SY[2], \ldots, SY[63] \rangle, \qquad (6.6)$$

the ideal in $\mathcal{P}$ generated by the 63 polynomials of the appendix. Then, in view of (6.4), the linear space $\mathcal{P}/\mathcal{S}$ has the generating function

$$F := \frac{1}{D}\left(U + \frac{UX_1}{1-X_1} + \frac{UX_2}{1-X_2} + \frac{V}{1-X_1} + \frac{W}{1-X_2}\right), \qquad (6.7)$$

where

$$D := \prod_{i=1}^{9}(1-K_i), \qquad U := 1 + \sum_{i=1}^{3} U_i, \qquad V := \sum_{i=1}^{4} V_i, \qquad W := \sum_{i=1}^{4} W_i, \qquad (6.8)$$

and we have set $U_3 := U_1U_2$. This is a generating function in the sense that the summands appearing in the formal expansion of F constitute a basis of the linear space $\mathcal{P}/\mathcal{S}$.

Replacing each invariant I in (6.7) by $q^{\deg(I)}$ leads directly to the formula (3.4) for the Molien series. More strikingly, if each invariant I is replaced by $a^{m_a}s^{m_s}p^{m_p}b^{m_b}$, where m_a, m_s, m_p and m_b are the degrees of I in the components of t, $\mathbf{s}$, $\mathbf{p}$ and $\mathbf{b}$, respectively, then (6.7) reduces immediately to the expression (3.7) for the generating function $G(a, s, p, b)$.

These observations imply that no further syzygies are required. More precisely, it follows that: (i) $\ker\phi = \mathcal{S}$; (ii) the 21 invariants of table 1 do indeed constitute a fundamental set of invariants; and (iii) the summands in the formal expansion of F provide a linear basis of $\mathcal{R}^G$. Hence, we have

Theorem 6.1. *Let V be the vector space spanned by the 16 elements of the two-qubit density matrix ρ. When expressed in terms of the fundamental invariants of table 1, the ring $\mathcal{R}^G$ of polynomials in the elements of V that are invariant under the local action of $G = GL_A(2) \times GL_B(2)$ has the structure*

$$\begin{aligned}\mathcal{R}^G = (1 \oplus U_1 \oplus U_2 \oplus U_1U_2)\mathbb{C}[K_1, K_2, \ldots, K_9] \\ \oplus ((1 \oplus U_1 \oplus U_2 \oplus U_1U_2)X_1 \\ \oplus V_1 \oplus V_2 \oplus V_3 \oplus V_4)\mathbb{C}[X_1, K_1, K_2, \ldots, K_9] \\ \oplus ((1 \oplus U_1 \oplus U_2 \oplus U_1U_2)X_2 \\ \oplus W_1 \oplus W_2 \oplus W_3 \oplus W_4)\mathbb{C}[X_2, K_1, K_2, \ldots, K_9].\end{aligned} \tag{6.9}$$

7. Primary and secondary invariants

The next task is to determine sets of primary and secondary invariants, and thereby express $\mathcal{R}^G$ in the Hironaka decomposition form (1.3). Comparison of the general form (1.4) of the Molien series with the explicit form (3.4) reveals that there are precisely ten algebraically independent invariants, with respect to which $\mathcal{R}^G$ is free. From (6.9), we have 11 candidates, namely $X_1, X_2, K_1, \ldots, K_9$. However, X_1 and X_2 are not algebraically independent of one another and $K_1, K_2, \ldots, K_9$. This is so because there exists a syzygy, still of the first kind, linking X_1, X_2 and all these K_i. It is obtained by finding an element $SX \in \mathcal{P}$ in the subideal $\langle SY[1], SY[2], SY[43]\rangle$ of $\mathcal{S}$, which is a polynomial in $X_1, X_2, K_1, \ldots, K_9$, and does not contain U_1 and U_2. This has been done using Maple. The polynomial SX is about ten pages long and is of degree 48. Its dependence on X_1 and X_2 is illustrated by setting $K_i = z^{\deg(K_i)}$ for $i = 1, 2, \ldots, 9$, which gives

$$\begin{aligned}SX|_{K_i=z^{\deg(K_i)}} = 16X_1^4X_2^4 + 8832z^{36}X_1X_2 - 7040X_1^2z^{30}X_2 - 2112z^{30}X_2^3 + 144z^{24}X_1^4 \\ - 2112z^{30}X_1^3 - 1088z^{42}X_1 + 1984X_1^2z^{24}X_2^2 + 128z^{24}X_2^3X_1 - 1088z^{42}X_2 \\ + 144z^{24}X_2^4 + 896X_1^3z^{18}X_2^2 + 6496z^{36}X_2^2 - 7040z^{30}X_2^2X_1 - 32z^{12}X_1^4X_2^2 \\ - 64X_1^4z^6X_2^3 + 192z^{18}X_1^4X_2 + 128X_1^3z^{24}X_2 - 32z^{12}X_2^4X_1^2 + 896X_1^2z^{18}X_2^3 \\ - 6512z^{48} + 192z^{18}X_2^4X_1 - 384X_1^3z^{12}X_2^3 + 6496z^{36}X_1^2 - 64z^6X_2^4X_1^3.\end{aligned} \tag{7.1}$$

Because $SX \in \mathcal{S}$, under the map ϕ we obtain the syzygy $\phi(SX) = 0$. Its existence confirms that X_1 and X_2 cannot both be primary invariants alongside K_i for $i = 1, 2, \ldots, 9$.

However, in our expansion of F in (6.7) it is clear that arbitrarily large powers of both X_1 and X_2 occur. The way out of this apparent impasse is to note first that (6.9) can be recast in the form

$$\begin{aligned}\mathcal{R}^G &= (U(1 \oplus X_1\mathbb{C}[X_1] \oplus X_2\mathbb{C}[X_2]) \oplus V\mathbb{C}[X_1] \oplus W\mathbb{C}[X_2])\mathbb{C}[K_1, K_2, \ldots, K_9] \\ &= \left(U\frac{\mathbb{C}[X_1, X_2]}{\langle X_1X_2\rangle} \oplus V\frac{\mathbb{C}[X_1, X_2]}{\langle X_2\rangle} \oplus W\frac{\mathbb{C}[X_1, X_2]}{\langle X_1\rangle}\right)\mathbb{C}[K_1, K_2, \ldots, K_9],\end{aligned} \tag{7.2}$$

where $U = 1 \oplus U_1 \oplus U_2 \oplus U_1U_2$, $V = V_1 \oplus V_2 \oplus V_3 \oplus V_4$ and $W = W_1 \oplus W_2 \oplus W_3 \oplus W_4$. In each of the quotients appearing in (7.2), $\langle Z\rangle$ denotes the ideal in $\mathbb{C}[X_1, X_2]$ generated by Z. If we now set $X = (X_1 + X_2)$ and $Y = (X_1 - X_2)$, so that $X_1 = (X + Y)/2$, $X_2 = (X - Y)/2$ and $X_1X_2 = (X^2 - Y^2)/4$, then

$$\frac{\mathbb{C}[X_1, X_2]}{\langle X_1X_2\rangle} = \frac{\mathbb{C}[X, Y]}{\langle X^2 - Y^2\rangle} = \mathbb{C}[X](1 \oplus Y); \tag{7.3}$$

$$\frac{\mathbb{C}[X_1, X_2]}{\langle X_2\rangle} = \frac{\mathbb{C}[X, Y]}{\langle X - Y\rangle} = \mathbb{C}[X]; \tag{7.4}$$

$$\frac{\mathbb{C}[X_1, X_2]}{\langle X_1\rangle} = \frac{\mathbb{C}[X, Y]}{\langle X + Y\rangle} = \mathbb{C}[X]. \tag{7.5}$$

It follows that

$$\mathcal{R}^G = (U \oplus UY \oplus V \oplus W)\mathbb{C}[K_1, K_2, \ldots, K_9, X]. \tag{7.6}$$

This not only demonstrates explicitly that $\mathcal{R}^G$ is Cohen–Macaulay, but also that it has a Hironaka decomposition (1.3) with 10 primary invariants $K_1, K_2, \ldots, K_9, X$, and 15 secondary invariants coming from $U \oplus UY \oplus V \oplus W$.

Adopting a more uniform notation, it follows that our primary and secondary invariants and their degrees can be identified as follows—the 10 primary invariants:

$$\begin{array}{ll} K_1 & \text{deg } 1; \\ K_2, K_3, K_4 & \text{deg } 2; \\ K_5, K_6 & \text{deg } 3; \\ K_7, K_8, K_9 & \text{deg } 4; \\ K_{10} := X_1 + X_2 & \text{deg } 6, \end{array} \tag{7.7}$$

and the 15 secondary invariants:

$$\begin{array}{llllll} J_1 := U_1 & \text{deg } 4; & J_8 := V_1 & \text{deg } 6; & J_9 := W_1 & \text{deg } 6; \\ J_2 := U_2 & \text{deg } 5; & J_{10} := V_2 & \text{deg } 7; & J_{11} := W_2 & \text{deg } 7; \\ J_3 := X_1 - X_2 & \text{deg } 6; & J_{12} := V_3 & \text{deg } 8; & J_{13} := W_3 & \text{deg } 8; \\ J_4 := J_1 J_2 & \text{deg } 9; & J_{14} := V_4 & \text{deg } 9; & J_{15} := W_4 & \text{deg } 9; \\ J_5 := J_1 J_3 & \text{deg } 10; & & & & \\ J_6 := J_2 J_3 & \text{deg } 11; & & & & \\ J_7 := J_1 J_2 J_3 & \text{deg } 15. & & & & \end{array} \tag{7.8}$$

8. Conclusion

We have arrived at the following result.

Theorem 8.1. *Let V be the vector space spanned by the 16 elements of the two-qubit density matrix ρ. Then, the ring $\mathcal{R}^G$ of polynomials in the elements of V that are invariant under the local action of $G = GL_A(2) \times GL_B(2)$ is Cohen–Macaulay, and its structure is given by*

$$\mathcal{R}^G = \oplus_{k=0}^{15} J_k \mathbb{C}[K_1, K_2, \ldots, K_{10}], \tag{8.1}$$

where $J_0 = 1$, and the remaining invariants are defined in (7.7) and (7.8) in terms of the fundamental invariants listed in table 1.

With 10 primary invariants K_i of degrees 1, 2, 2, 2, 3, 3, 4, 4, 4, 6, and 15 secondary invariants J_k of degrees 4, 5, 6, 6, 6, 7, 7, 8, 8, 9, 9, 9, 10, 11, 15, it follows that the Molien series for $\mathcal{R}^G$ is given by the $|q| < 1$ expansion of

$$M(q) = \frac{\sum_{k=0}^{15} q^{\deg J_k}}{\prod_{i=1}^{10} (1 - q^{\deg K_i})} = \frac{1 + q^4 + q^5 + 3q^6 + 2q^7 + 2q^8 + 3q^9 + q^{10} + q^{11} + q^{15}}{(1-q)(1-q^2)^3(1-q^3)^2(1-q^4)^3(1-q^6)}. \tag{8.2}$$

It might also be noticed that, treated as a rational function of q, this Molien series satisfies the identity

$$M(1/q) = q^{16} M(q), \tag{8.3}$$

where 16 is the dimension of the vector space V spanned by the elements of the two-qubit density matrix ρ as exhibited, for example, in (1.5). This is sufficient to show that the Cohen–Macaulay ring $\mathcal{R}^G$ is also Gorenstein [11].

While the methods employed here in the mixed two-qubit case may be used more generally in the mixed N-qubit case, the calculations do not appear to be tractable at present even for $N = 3$, let alone for $N > 3$.

Acknowledgments

It is a pleasure to acknowledge helpful comments from Chris Cummins, Peter Bouwknegt and Amnon Neeman. In addition, RCK is grateful for the award of Leverhulme Emeritus Fellowship in support of this work.

Appendix

A computer search reveals the existence of the following set of syzygies of the first kind. In each case, the polynomial expression $SY[k]$ for $k = 1, 2, \ldots, 63$ is identically zero when expanded in terms of the matrix elements of ρ parameterized as in (1.5), that is to say $\phi(SY[k]) = 0$ for $k = 1, 2, \ldots, 63$. Each $SY[k]$ is homogeneous in the sense that for fixed k, the image under ϕ of each summand of $SY[k]$ is homogeneous and of the same degree. Setting $SY[k] = 0$ allows the first term in each expression to be eliminated in favour of those terms that follow, all of which appear in the expansion of F as defined in (6.7). Of course, by virtue of their elimination, the first terms themselves do not appear in this expansion.

The notation is such that subscripts k in the invariants of table 1 have been displayed as $[k]$ in the verbatim Maple output. The only other change is the use of $KK[5] = \det b$ rather than $K[5] = 6 \det b$.

```
# elimination of terms U[1]^2 and U[2]^2

SY[1] := U[1]^2 − 2 ∗ K[2]^2 ∗ K[3] ∗ K[4] + 4 ∗ K[2] ∗ K[3] ∗ K[9] + 4 ∗ K[2] ∗ K[4] ∗ K[8]
− 4 ∗ K[2] ∗ K[6]^2 + 2 ∗ K[3] ∗ K[4] ∗ K[7] + 8 ∗ U[2] ∗ K[6] − 4 ∗ X[1] ∗ K[4]
− 4 ∗ X[2] ∗ K[3] − 4 ∗ K[8] ∗ K[9] :

SY[2] := 2 ∗ U[2]^2 − K[2]^2 ∗ K[6]^2 − 2 ∗ K[3] ∗ K[4] ∗ KK[5]^2 + 2 ∗ U[1] ∗ KK[5] ∗ K[6]
+ 2 ∗ K[2] ∗ K[8] ∗ K[9] + K[6]^2 ∗ K[7] − 2 ∗ X[1] ∗ K[9] − 2 ∗ X[2] ∗ K[8] :

# elimination of all terms U[1] ∗ V[k] and U[1] ∗ W[k]

SY[3] := W[1] ∗ U[1] − 2 ∗ V[1] ∗ K[8] + 2 ∗ V[2] ∗ K[6] − 2 ∗ W[3] ∗ K[3] :

SY[4] := V[1] ∗ U[1] − 2 ∗ V[3] ∗ K[4] − 2 ∗ W[1] ∗ K[9] + 2 ∗ W[2] ∗ K[6] :

SY[5] := W[2] ∗ U[1] + 2 ∗ V[1] ∗ K[2] ∗ K[6] − 2 ∗ W[1] ∗ K[4] ∗ KK[5] − 2 ∗ V[2] ∗ K[9]
+ 4 ∗ W[3] ∗ K[6] − 2 ∗ W[4] ∗ K[3] :

SY[6] := V[2] ∗ U[1] − 2 ∗ V[1] ∗ K[3] ∗ KK[5] + 2 ∗ W[1] ∗ K[2] ∗ K[6] + 4 ∗ V[3] ∗ K[6]
− 2 ∗ V[4] ∗ K[4] − 2 ∗ W[2] ∗ K[8] :

SY[7] := V[3] ∗ U[1] − V[1] ∗ K[2]^2 ∗ K[3] + 2 ∗ V[1] ∗ K[2] ∗ K[8] + V[1] ∗ K[3] ∗ K[7]
+ 2 ∗ W[1] ∗ KK[5] ∗ K[6] − 2 ∗ W[2] ∗ K[3] ∗ KK[5] − 2 ∗ V[1] ∗ X[1] :

SY[8] := W[3] ∗ U[1] − W[1] ∗ K[2]^2 ∗ K[4] + 2 ∗ W[1] ∗ K[2] ∗ K[9] + W[1] ∗ K[4] ∗ K[7]
+ 2 ∗ V[1] ∗ KK[5] ∗ K[6] − 2 ∗ V[2] ∗ K[4] ∗ KK[5] − 2 ∗ W[1] ∗ X[2] :

SY[9] := V[4] ∗ U[1] − V[2] ∗ K[2]^2 ∗ K[3] − 2 ∗ W[1] ∗ K[2] ∗ K[3] ∗ KK[5]
+ 2 ∗ V[2] ∗ K[2] ∗ K[8] + V[2] ∗ K[3] ∗ K[7] − 2 ∗ V[3] ∗ K[3] ∗ KK[5]
+ 2 ∗ W[1] ∗ KK[5] ∗ K[8] − 2 ∗ V[2] ∗ X[1] :
```

```
SY[10] := W[4]*U[1] - 2*V[1]*K[2]*K[4]*KK[5] - W[2]*K[2]^2*K[4]
+ 2*V[1]*KK[5]*K[9] + 2*W[2]*K[2]*K[9] + W[2]*K[4]*K[7]
- 2*W[3]*K[4]*KK[5] - 2*W[2]*X[2] :
# elimination of all terms U[2]*V[k] and U[2]*W[k]
SY[11] := W[1]*U[2] - V[1]*K[3]*KK[5] + V[3]*K[6] - W[2]*K[8] :
SY[12] := W[2]*U[2] + V[1]*KK[5]*K[6] - V[3]*K[9] - W[1]*X[2] :

SY[13] := 2*V[3]*U[2] + W[1]*K[2]^2*K[6] - 2*V[1]*KK[5]*K[8]
+ 4*V[2]*KK[5]*K[6] - W[1]*K[6]*K[7] - 2*W[3]*K[3]*KK[5] - 2*V[4]*K[9] :

SY[14] := 2*V[4]*U[2] - W[1]*K[2]^2*K[8] + 2*W[1]*K[3]*KK[5]^2
- 2*V[2]*KK[5]*K[8] + W[1]*K[7]*K[8] - 2*V[3]*X[1] :
SY[15] := V[1]*U[2] - W[1]*K[4]*KK[5] - V[2]*K[9] + W[3]*K[6] :
SY[16] := V[2]*U[2] + W[1]*KK[5]*K[6] - V[1]*X[1] - W[3]*K[8] :

SY[17] := 2*W[3]*U[2] + V[1]*K[2]^2*K[6] - V[1]*K[6]*K[7]
- 2*V[3]*K[4]*KK[5] - 2*W[1]*KK[5]*K[9] + 4*W[2]*KK[5]*K[6]
- 2*W[4]*K[8] :

SY[18] := 2*W[4]*U[2] - V[1]*K[2]^2*K[9] + 2*V[1]*K[4]*KK[5]^2
+ V[1]*K[7]*K[9] - 2*W[2]*KK[5]*K[9] - 2*W[3]*X[2] :
# elimination of all terms (V[1], V[2], W[3], W[4])*(V[3], V[4], W[1], W[2])

SY[19] := 2*V[1]*W[1] + 2*U[2]*K[3]*K[4]*KK[5] - 2*K[3]*KK[5]*K[6]*K[9]
- 2*K[4]*KK[5]*K[6]*K[8] + 2*KK[5]*K[6]^3 - U[1]*U[2]*K[6] + U[1]*K[8]*K[9]:

SY[20] := 2*V[2]*W[1] - 2*U[2]*K[3]*KK[5]*K[6] + 2*X[1]*K[3]*K[4]*KK[5]
- 2*K[4]*KK[5]*K[8]^2 + 2*KK[5]*K[6]^2*K[8] + U[1]*U[2]*K[8]
- U[1]*X[1]*K[6] :

SY[21] := 4*W[1]*W[3] - 4*K[2]^2*K[3]*K[4]*KK[5]*K[6] + U[1]*K[2]^2*K[6]^2
+ 2*U[1]*K[3]*K[4]*KK[5]^2 + 8*K[2]*K[3]*KK[5]*K[6]*K[9]
+ 8*K[2]*K[4]*KK[5]*K[6]*K[8] - 8*K[2]*KK[5]*K[6]^3
+ 4*K[3]*K[4]*KK[5]*K[6]*K[7] - 4*U[2]*K[4]*KK[5]*K[8]
+ 12*U[2]*KK[5]*K[6]^2 - 2*U[1]*K[2]*K[8]*K[9] - U[1]*K[6]^2*K[7]
- 4*X[1]*K[4]*KK[5]*K[6] - 8*X[2]*K[3]*KK[5]*K[6]
- 4*KK[5]*K[6]*K[8]*K[9] + 2*U[1]*X[2]*K[8] :

SY[22] := 4*W[1]*W[4] - 2*K[2]^2*K[3]*K[4]*KK[5]*K[9]
- 4*K[2]^2*K[4]*KK[5]*K[6]^2 - 4*K[3]*K[4]^2*KK[5]^3
+ 4*U[2]*K[2]*K[4]*KK[5]*K[6] + U[1]*K[2]^2*K[6]*K[9]
+ 6*U[1]*K[4]*KK[5]^2*K[6] + 4*K[2]*K[3]*KK[5]*K[9]^2
```

```
+ 8 * K[2] * K[4] * KK[5] * K[8] * K[9] - 4 * K[2] * KK[5] * K[6]^2 * K[9]
+ 2 * K[3] * K[4] * KK[5] * K[7] * K[9] + 4 * K[4] * KK[5] * K[6]^2 * K[7]
- 2 * U[1] * U[2] * K[2] * K[9] + 12 * U[2] * KK[5] * K[6] * K[9] - U[1] * K[6] * K[7] * K[9]
- 8 * X[1] * K[4] * KK[5] * K[9] - 4 * X[2] * K[3] * KK[5] * K[9]
- 4 * X[2] * K[4] * KK[5] * K[8] - 4 * X[2] * KK[5] * K[6]^2 - 4 * KK[5] * K[8] * K[9]^2
+ 2 * U[1] * U[2] * X[2] :

SY[23] := 2 * V[1] * W[2] - 2 * U[2] * K[4] * KK[5] * K[6] + 2 * X[2] * K[3] * K[4] * KK[5]
- 2 * K[3] * KK[5] * K[9]^2 + 2 * KK[5] * K[6]^2 * K[9] + U[1] * U[2] * K[9]
- U[1] * X[2] * K[6] :

SY[24] := 2 * V[2] * W[2] - 4 * K[2]^2 * K[3] * K[4] * KK[5] * K[6] + U[1] * K[2]^2 * K[6]^2
+ 2 * U[2] * K[2] * K[3] * K[4] * KK[5] + 2 * U[1] * K[3] * K[4] * KK[5]^2
+ 6 * K[2] * K[3] * KK[5] * K[6] * K[9] + 6 * K[2] * K[4] * KK[5] * K[6] * K[8]
- 6 * K[2] * KK[5] * K[6]^3 + 4 * K[3] * K[4] * KK[5] * K[6] * K[7] - U[1] * U[2] * K[2] * K[6]
- 2 * U[2] * K[3] * KK[5] * K[9] - 2 * U[2] * K[4] * KK[5] * K[8] + 12 * U[2] * KK[5] * K[6]^2
- U[1] * K[2] * K[8] * K[9] - U[1] * K[6]^2 * K[7] - 6 * X[1] * K[4] * KK[5] * K[6]
- 6 * X[2] * K[3] * KK[5] * K[6] - 4 * KK[5] * K[6] * K[8] * K[9] + U[1] * X[1] * K[9]
+ U[1] * X[2] * K[8] :

SY[25] := 4 * W[2] * W[3] - 2 * K[2]^2 * K[3] * K[4] * KK[5] * K[9]
- 2 * K[2]^2 * K[4] * KK[5] * K[6]^2 + 4 * U[2] * K[2] * K[4] * KK[5] * K[6]
+ U[1] * K[2]^2 * K[6] * K[9] + 2 * U[1] * K[4] * KK[5]^2 * K[6] + 4 * K[2] * K[3] * KK[5] * K[9]^2
+ 4 * K[2] * K[4] * KK[5] * K[8] * K[9] - 4 * K[2] * KK[5] * K[6]^2 * K[9]
+ 2 * K[3] * K[4] * KK[5] * K[7] * K[9] + 2 * K[4] * KK[5] * K[6]^2 * K[7]
- 2 * U[1] * U[2] * K[2] * K[9] + 4 * U[2] * KK[5] * K[6] * K[9] - U[1] * K[6] * K[7] * K[9]
- 4 * X[1] * K[4] * KK[5] * K[9] - 4 * X[2] * K[3] * KK[5] * K[9]
- 4 * X[2] * K[4] * KK[5] * K[8] + 2 * U[1] * U[2] * X[2] :

SY[26] := 4 * W[2] * W[4] - 2 * K[2]^2 * K[4] * KK[5] * K[6] * K[9]
+ 4 * K[4]^2 * KK[5]^3 * K[6] + U[1] * K[2]^2 * K[9]^2 - 2 * U[1] * K[4] * KK[5]^2 * K[9]
+ 4 * X[2] * K[2] * K[4] * KK[5] * K[6] + 2 * K[4] * KK[5] * K[6] * K[7] * K[9]
- 4 * U[2] * X[2] * K[4] * KK[5] + 4 * U[2] * KK[5] * K[9]^2 - 2 * U[1] * X[2] * K[2] * K[9]
- U[1] * K[7] * K[9]^2 - 4 * X[2] * KK[5] * K[6] * K[9] + 2 * U[1] * X[2]^2 :

SY[27] := 4 * V[1] * V[3] - 4 * K[2]^2 * K[3] * K[4] * KK[5] * K[6] + U[1] * K[2]^2 * K[6]^2
+ 2 * U[1] * K[3] * K[4] * KK[5]^2 + 8 * K[2] * K[3] * KK[5] * K[6] * K[9]
+ 8 * K[2] * K[4] * KK[5] * K[6] * K[8] - 8 * K[2] * KK[5] * K[6]^3
+ 4 * K[3] * K[4] * KK[5] * K[6] * K[7] - 4 * U[2] * K[3] * KK[5] * K[9]
+ 12 * U[2] * KK[5] * K[6]^2 - 2 * U[1] * K[2] * K[8] * K[9] - U[1] * K[6]^2 * K[7]
- 8 * X[1] * K[4] * KK[5] * K[6] - 4 * X[2] * K[3] * KK[5] * K[6]
- 4 * KK[5] * K[6] * K[8] * K[9] + 2 * U[1] * X[1] * K[9] :
```

```
SY[28] := 4*V[2]*V[3] - 2*K[2]^2*K[3]*K[4]*KK[5]*K[8]
- 2*K[2]^2*K[3]*KK[5]*K[6]^2 + 4*U[2]*K[2]*K[3]*KK[5]*K[6]
+ U[1]*K[2]^2*K[6]*K[8] + 2*U[1]*K[3]*KK[5]^2*K[6]
+ 4*K[2]*K[3]*KK[5]*K[8]*K[9] + 4*K[2]*K[4]*KK[5]*K[8]^2
- 4*K[2]*KK[5]*K[6]^2*K[8] + 2*K[3]*K[4]*KK[5]*K[7]*K[8]
+ 2*K[3]*KK[5]*K[6]^2*K[7] - 2*U[1]*U[2]*K[2]*K[8]
+ 4*U[2]*KK[5]*K[6]*K[8] - U[1]*K[6]*K[7]*K[8] - 4*X[1]*K[3]*KK[5]*K[9]
- 4*X[1]*K[4]*KK[5]*K[8] - 4*X[2]*K[3]*KK[5]*K[8] + 2*U[1]*U[2]*X[1] :

SY[29] := 4*V[3]*W[3] + 2*U[2]*K[2]^2*K[3]*K[4]*KK[5] - 6*K[2]^2*KK[5]*K[6]^3
- 8*K[3]*K[4]*KK[5]^3*K[6] - U[1]*U[2]*K[2]^2*K[6]
- 4*U[2]*K[2]*K[3]*KK[5]*K[9] - 4*U[2]*K[2]*K[4]*KK[5]*K[8]
+ 8*U[2]*K[2]*KK[5]*K[6]^2 - 2*U[2]*K[3]*K[4]*KK[5]*K[7]
+ U[1]*K[2]^2*K[8]*K[9] - 2*U[1]*K[3]*KK[5]^2*K[9]
- 2*U[1]*K[4]*KK[5]^2*K[8] + 12*U[1]*KK[5]^2*K[6]^2
+ 8*K[2]*KK[5]*K[6]*K[8]*K[9] + 6*KK[5]*K[6]^3*K[7] + U[1]*U[2]*K[6]*K[7]
+ 4*U[2]*X[1]*K[4]*KK[5] + 4*U[2]*X[2]*K[3]*KK[5]
+ 8*U[2]*KK[5]*K[8]*K[9] - U[1]*K[7]*K[8]*K[9]
- 12*X[1]*KK[5]*K[6]*K[9] - 12*X[2]*KK[5]*K[6]*K[8] :

SY[30] := 4*V[3]*W[4] + 2*K[2]^3*K[4]*KK[5]*K[6]^2 + 4*K[2]*K[3]*K[4]^2*KK[5]^3
- 4*U[2]*K[2]^2*K[4]*KK[5]*K[6] - 4*U[1]*K[2]*K[4]*KK[5]^2*K[6]
+ 2*X[2]*K[2]^2*K[3]*K[4]*KK[5] - 2*K[2]^2*K[4]*KK[5]*K[8]*K[9]
+ 2*K[2]^2*KK[5]*K[6]^2*K[9] - 2*K[2]*K[4]*KK[5]*K[6]^2*K[7]
- 4*K[4]^2*KK[5]^3*K[8] + 8*K[4]*KK[5]^3*K[6]^2 + U[1]*U[2]*K[2]^2*K[9]
+ 2*U[2]*U[1]*K[4]*KK[5]^2 - 4*U[2]*K[2]*KK[5]*K[6]*K[9]
+ 4*U[2]*K[4]*KK[5]*K[6]*K[7] - U[1]*X[2]*K[2]^2*K[6]
- 6*U[1]*KK[5]^2*K[6]*K[9] + 4*X[1]*K[2]*K[4]*KK[5]*K[9]
- 4*X[2]*K[2]*K[3]*KK[5]*K[9] + 8*X[2]*K[2]*KK[5]*K[6]^2
- 2*X[2]*K[3]*K[4]*KK[5]*K[7] - 4*K[2]*KK[5]*K[8]*K[9]^2
- 2*K[4]*KK[5]*K[7]*K[8]*K[9] - 2*KK[5]*K[6]^2*K[7]*K[9]
- U[1]*U[2]*K[7]*K[9] - 12*U[2]*X[2]*KK[5]*K[6]
+ U[1]*X[2]*K[6]*K[7] + 4*X[1]*KK[5]*K[9]^2 + 4*X[2]^2*K[3]*KK[5]
+ 8*X[2]*KK[5]*K[8]*K[9] :

SY[31] := 4*V[1]*V[4] - 2*K[2]^2*K[3]*K[4]*KK[5]*K[8]
- 4*K[2]^2*K[3]*KK[5]*K[6]^2 - 4*K[3]^2*K[4]*KK[5]^3
+ 4*U[2]*K[2]*K[3]*KK[5]*K[6] + U[1]*K[2]^2*K[6]*K[8]
+ 6*U[1]*K[3]*KK[5]^2*K[6] + 8*K[2]*K[3]*KK[5]*K[8]*K[9]
+ 4*K[2]*K[4]*KK[5]*K[8]^2 - 4*K[2]*KK[5]*K[6]^2*K[8]
+ 2*K[3]*K[4]*KK[5]*K[7]*K[8] + 4*K[3]*KK[5]*K[6]^2*K[7]
```

− 2 ∗ U[1] ∗ U[2] ∗ K[2] ∗ K[8] + 12 ∗ U[2] ∗ KK[5] ∗ K[6] ∗ K[8] − U[1] ∗ K[6] ∗ K[7] ∗ K[8]
− 4 ∗ X[1] ∗ K[3] ∗ KK[5] ∗ K[9] − 4 ∗ X[1] ∗ K[4] ∗ KK[5] ∗ K[8] − 4 ∗ X[1] ∗ KK[5] ∗ K[6]^2
− 8 ∗ X[2] ∗ K[3] ∗ KK[5] ∗ K[8] − 4 ∗ KK[5] ∗ K[8]^2 ∗ K[9] + 2 ∗ U[1] ∗ U[2] ∗ X[1] :

SY[32] := 4 ∗ V[2] ∗ V[4] − 2 ∗ K[2]^2 ∗ K[3] ∗ KK[5] ∗ K[6] ∗ K[8]
+ 4 ∗ K[3]^2 ∗ KK[5]^3 ∗ K[6] + U[1] ∗ K[2]^2 ∗ K[8]^2 − 2 ∗ U[1] ∗ K[3] ∗ KK[5]^2 ∗ K[8]
+ 4 ∗ X[1] ∗ K[2] ∗ K[3] ∗ KK[5] ∗ K[6] + 2 ∗ K[3] ∗ KK[5] ∗ K[6] ∗ K[7] ∗ K[8]
− 4 ∗ U[2] ∗ X[1] ∗ K[3] ∗ KK[5] + 4 ∗ U[2] ∗ KK[5] ∗ K[8]^2 − 2 ∗ U[1] ∗ X[1] ∗ K[2] ∗ K[8]
− U[1] ∗ K[7] ∗ K[8]^2 − 4 ∗ X[1] ∗ KK[5] ∗ K[6] ∗ K[8] + 2 ∗ U[1] ∗ X[1]^2 :

SY[33] := 4 ∗ V[4] ∗ W[3] + 2 ∗ K[2]^3 ∗ K[3] ∗ KK[5] ∗ K[6]^2
+ 4 ∗ K[2] ∗ K[3]^2 ∗ K[4] ∗ KK[5]^3 − 4 ∗ U[2] ∗ K[2]^2 ∗ K[3] ∗ KK[5] ∗ K[6]
− 4 ∗ U[1] ∗ K[2] ∗ K[3] ∗ KK[5]^2 ∗ K[6] + 2 ∗ X[1] ∗ K[2]^2 ∗ K[3] ∗ K[4] ∗ KK[5]
− 2 ∗ K[2]^2 ∗ K[3] ∗ KK[5] ∗ K[8] ∗ K[9] + 2 ∗ K[2]^2 ∗ KK[5] ∗ K[6]^2 ∗ K[8]
− 2 ∗ K[2] ∗ K[3] ∗ KK[5] ∗ K[6]^2 ∗ K[7] − 4 ∗ K[3]^2 ∗ KK[5]^3 ∗ K[9]
+ 8 ∗ K[3] ∗ KK[5]^3 ∗ K[6]^2 + U[1] ∗ U[2] ∗ K[2]^2 ∗ K[8] + 2 ∗ U[2] ∗ U[1] ∗ K[3] ∗ KK[5]^2
− 4 ∗ U[2] ∗ K[2] ∗ KK[5] ∗ K[6] ∗ K[8] + 4 ∗ U[2] ∗ K[3] ∗ KK[5] ∗ K[6] ∗ K[7]
− U[1] ∗ X[1] ∗ K[2]^2 ∗ K[6] − 6 ∗ U[1] ∗ KK[5]^2 ∗ K[6] ∗ K[8]
− 4 ∗ X[1] ∗ K[2] ∗ K[4] ∗ KK[5] ∗ K[8] + 8 ∗ X[1] ∗ K[2] ∗ KK[5] ∗ K[6]^2
− 2 ∗ X[1] ∗ K[3] ∗ K[4] ∗ KK[5] ∗ K[7] + 4 ∗ X[2] ∗ K[2] ∗ K[3] ∗ KK[5] ∗ K[8]
− 4 ∗ K[2] ∗ KK[5] ∗ K[8]^2 ∗ K[9] − 2 ∗ K[3] ∗ KK[5] ∗ K[7] ∗ K[8] ∗ K[9]
− 2 ∗ KK[5] ∗ K[6]^2 ∗ K[7] ∗ K[8] − U[1] ∗ U[2] ∗ K[7] ∗ K[8]
− 12 ∗ U[2] ∗ X[1] ∗ KK[5] ∗ K[6] + U[1] ∗ X[1] ∗ K[6] ∗ K[7]
+ 4 ∗ X[1]^2 ∗ K[4] ∗ KK[5] + 8 ∗ X[1] ∗ KK[5] ∗ K[8] ∗ K[9] + 4 ∗ X[2] ∗ KK[5] ∗ K[8]^2 :

SY[34] := 4 ∗ V[4] ∗ W[4] − 4 ∗ K[2]^4 ∗ K[3] ∗ K[4] ∗ KK[5] ∗ K[6]
+ 2 ∗ U[2] ∗ K[2]^3 ∗ K[3] ∗ K[4] ∗ KK[5] + U[1] ∗ K[2]^4 ∗ K[6]^2
+ 2 ∗ U[1] ∗ K[2]^2 ∗ K[3] ∗ K[4] ∗ KK[5]^2 + 8 ∗ K[2]^3 ∗ K[3] ∗ KK[5] ∗ K[6] ∗ K[9]
+ 8 ∗ K[2]^3 ∗ K[4] ∗ KK[5] ∗ K[6] ∗ K[8] − 18 ∗ K[2]^3 ∗ KK[5] ∗ K[6]^3
+ 8 ∗ K[2]^2 ∗ K[3] ∗ K[4] ∗ KK[5] ∗ K[6] ∗ K[7] − 16 ∗ K[2] ∗ K[3] ∗ K[4] ∗ KK[5]^3 ∗ K[6]
− U[1] ∗ U[2] ∗ K[2]^3 ∗ K[6] − 4 ∗ U[2] ∗ K[2]^2 ∗ K[3] ∗ KK[5] ∗ K[9]
− 4 ∗ U[2] ∗ K[2]^2 ∗ K[4] ∗ KK[5] ∗ K[8] + 28 ∗ U[2] ∗ K[2]^2 ∗ KK[5] ∗ K[6]^2
− 2 ∗ U[2] ∗ K[2] ∗ K[3] ∗ K[4] ∗ KK[5] ∗ K[7] + 4 ∗ U[2] ∗ K[3] ∗ K[4] ∗ KK[5]^3
− U[1] ∗ K[2]^3 ∗ K[8] ∗ K[9] − 2 ∗ U[1] ∗ K[2]^2 ∗ K[6]^2 ∗ K[7]
− 2 ∗ U[1] ∗ K[2] ∗ K[3] ∗ KK[5]^2 ∗ K[9] − 2 ∗ U[1] ∗ K[2] ∗ K[4] ∗ KK[5]^2 ∗ K[8]
+ 20 ∗ U[1] ∗ K[2] ∗ KK[5]^2 ∗ K[6]^2 − 2 ∗ U[1] ∗ K[3] ∗ K[4] ∗ KK[5]^2 ∗ K[7]
− 8 ∗ X[1] ∗ K[2]^2 ∗ K[4] ∗ KK[5] ∗ K[6] − 8 ∗ X[2] ∗ K[2]^2 ∗ K[3] ∗ KK[5] ∗ K[6]
+ 8 ∗ K[2]^2 ∗ KK[5] ∗ K[6] ∗ K[8] ∗ K[9] − 8 ∗ K[2] ∗ K[3] ∗ KK[5] ∗ K[6] ∗ K[7] ∗ K[9]
− 8 ∗ K[2] ∗ K[4] ∗ KK[5] ∗ K[6] ∗ K[7] ∗ K[8] + 18 ∗ K[2] ∗ KK[5] ∗ K[6]^3 ∗ K[7]
− 4 ∗ K[3] ∗ K[4] ∗ KK[5] ∗ K[6] ∗ K[7]^2 + 4 ∗ KK[5]^3 ∗ K[6]^3
+ U[1] ∗ U[2] ∗ K[2] ∗ K[6] ∗ K[7] − 14 ∗ U[2] ∗ U[1] ∗ KK[5]^2 ∗ K[6]

```
+ 4 * U[2] * X[1] * K[2] * K[4] * KK[5] + 4 * U[2] * X[2] * K[2] * K[3] * KK[5]
− 20 * U[2] * KK[5] * K[6]^2 * K[7] + U[1] * X[1] * K[2]^2 * K[9]
+ 2 * U[1] * X[1] * K[4] * KK[5]^2 + U[1] * X[2] * K[2]^2 * K[8]
+ 2 * U[1] * X[2] * K[3] * KK[5]^2 + U[1] * K[2] * K[7] * K[8] * K[9]
+ 2 * U[1] * KK[5]^2 * K[8] * K[9] + U[1] * K[6]^2 * K[7]^2
− 16 * X[1] * K[2] * KK[5] * K[6] * K[9] + 8 * X[1] * K[4] * KK[5] * K[6] * K[7]
− 16 * X[2] * K[2] * KK[5] * K[6] * K[8] + 8 * X[2] * K[3] * KK[5] * K[6] * K[7]
+ 8 * KK[5] * K[6] * K[7] * K[8] * K[9] + 4 * U[2] * X[1] * KK[5] * K[9]
+ 4 * U[2] * X[2] * KK[5] * K[8] − U[1] * X[1] * K[7] * K[9] − U[1] * X[2] * K[7] * K[8] :

# elimination of all terms W[k] * X[1]

SY[35] := W[1] * X[1] − V[2] * K[3] * KK[5] − W[1] * K[2] * K[8] − V[3] * K[8] + V[4] * K[6] :

SY[36] := W[2] * X[1] − V[1] * K[2] * K[3] * KK[5] + W[1] * K[2]^2 * K[6]
− V[1] * KK[5] * K[8] + 3 * V[2] * KK[5] * K[6] + V[3] * K[2] * K[6] − W[1] * K[6] * K[7]
− W[2] * K[2] * K[8] − 2 * W[3] * K[3] * KK[5] − V[4] * K[9] :

SY[37] := 2 * W[3] * X[1] − V[1] * K[2]^2 * K[8] − 2 * V[1] * K[3] * KK[5]^2
+ V[2] * K[2]^2 * K[6] + 4 * W[1] * K[2] * KK[5] * K[6] + V[1] * K[7] * K[8]
− V[2] * K[6] * K[7] + 6 * V[3] * KK[5] * K[6] − 2 * V[4] * K[4] * KK[5]
− 4 * W[2] * KK[5] * K[8] − 2 * W[3] * K[2] * K[8] :

SY[38] := 2 * W[4] * X[1] + V[1] * K[2]^3 * K[6] − 2 * W[1] * K[2]^2 * K[4] * KK[5]
− V[1] * K[2] * K[6] * K[7] + 6 * V[1] * KK[5]^2 * K[6] − V[2] * K[2]^2 * K[9]
− 2 * V[2] * K[4] * KK[5]^2 − 2 * V[3] * K[2] * K[4] * KK[5]
+ 2 * W[1] * K[4] * KK[5] * K[7] + 4 * W[2] * K[2] * KK[5] * K[6]
+ 2 * W[3] * K[2]^2 * K[6] + V[2] * K[7] * K[9] − 2 * V[3] * KK[5] * K[9]
− 4 * W[1] * X[2] * KK[5] − 2 * W[3] * K[6] * K[7] − 2 * W[4] * K[2] * K[8] :

# elimination of all terms V[k] * X[2]

SY[39] := V[1] * X[2] − V[1] * K[2] * K[9] − W[2] * K[4] * KK[5] −W[3] * K[9] + W[4] * K[6] :

SY[40] := V[2] * X[2] + V[1] * K[2]^2 * K[6] − W[1] * K[2] * K[4] * KK[5] − V[1] * K[6] * K[7]
− V[2] * K[2] * K[9] − 2 * V[3] * K[4] * KK[5] − W[1] * KK[5] * K[9]
+ 3 * W[2] * KK[5] * K[6] + W[3] * K[2] * K[6] − W[4] * K[8] :

SY[41] := 2 * V[3] * X[2] + 4 * V[1] * K[2] * KK[5] * K[6] − W[1] * K[2]^2 * K[9]
− 2 * W[1] * K[4] * KK[5]^2 + W[2] * K[2]^2 * K[6] − 4 * V[2] * KK[5] * K[9]
− 2 * V[3] * K[2] * K[9] + W[1] * K[7] * K[9] − W[2] * K[6] * K[7] + 6 * W[3] * KK[5] * K[6]
− 2 * W[4] * K[3] * KK[5] :

SY[42] := 2 * V[4] * X[2] − 2 * V[1] * K[2]^2 * K[3] * KK[5] + W[1] * K[2]^3 * K[6]
+ 2 * V[1] * K[3] * KK[5] * K[7] + 4 * V[2] * K[2] * KK[5] * K[6] + 2 * V[3] * K[2]^2 * K[6]
```

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− W[1] ∗ K[2] ∗ K[6] ∗ K[7] + 6 ∗ W[1] ∗ KK[5]^2 ∗ K[6] − W[2] ∗ K[2]^2 ∗ K[8]
− 2 ∗ W[2] ∗ K[3] ∗ KK[5]^2 − 2 ∗ W[3] ∗ K[2] ∗ K[3] ∗ KK[5] − 4 ∗ V[1] ∗ X[1] ∗ KK[5]
− 2 ∗ V[3] ∗ K[6] ∗ K[7] − 2 ∗ V[4] ∗ K[2] ∗ K[9] + W[2] ∗ K[7] ∗ K[8]
− 2 ∗ W[3] ∗ KK[5] ∗ K[8] :

# elimination of X[1] ∗ X[2]

SY[43] := 2 ∗ X[1] ∗ X[2] − K[2]^3 ∗ K[6]^2 − 2 ∗ K[2] ∗ K[3] ∗ K[4] ∗ KK[5]^2
+ 2 ∗ U[2] ∗ K[2]^2 ∗ K[6] + 2 ∗ U[1] ∗ K[2] ∗ KK[5] ∗ K[6] + K[2]^2 ∗ K[8] ∗ K[9]
+ K[2] ∗ K[6]^2 ∗ K[7] + 2 ∗ K[3] ∗ KK[5]^2 ∗ K[9] + 2 ∗ K[4] ∗ KK[5]^2 ∗ K[8]
− 2 ∗ KK[5]^2 ∗ K[6]^2 − 2 ∗ U[1] ∗ U[2] ∗ KK[5] − 2 ∗ U[2] ∗ K[6] ∗ K[7]
− 2 ∗ X[1] ∗ K[2] ∗ K[9] − 2 ∗ X[2] ∗ K[2] ∗ K[8] + K[7] ∗ K[8] ∗ K[9] :

# elimination of all terms (V[3], V[4], W[1], W[2]) ∗ (V[3], V[4], W[1], W[2])
# and all terms (V[1], V[2], W[3], W[4]) ∗ (V[1], V[2], W[3], W[4])

SY[44] := 2 ∗ W[1]^2 + K[2]^2 ∗ K[3] ∗ K[6]^2 + 2 ∗ K[3]^2 ∗ K[4] ∗ KK[5]^2
− 2 ∗ U[1] ∗ K[3] ∗ KK[5] ∗ K[6] − 2 ∗ K[2] ∗ K[3] ∗ K[8] ∗ K[9] − K[3] ∗ K[6]^2 ∗ K[7]
− 4 ∗ U[2] ∗ K[6] ∗ K[8] + 2 ∗ X[1] ∗ K[6]^2 + 2 ∗ X[2] ∗ K[3] ∗ K[8] + 2 ∗ K[8]^2 ∗ K[9] :

SY[45] := 2 ∗ W[1] ∗ W[2] + K[2]^2 ∗ K[3] ∗ K[6] ∗ K[9] − 2 ∗ K[2]^2 ∗ K[6]^3
− 2 ∗ K[3] ∗ K[4] ∗ KK[5]^2 ∗ K[6] − 2 ∗ U[2] ∗ K[2] ∗ K[3] ∗ K[9] + 2 ∗ U[2] ∗ K[2] ∗ K[6]^2
− U[1] ∗ K[3] ∗ KK[5] ∗ K[9] + 3 ∗ U[1] ∗ KK[5] ∗ K[6]^2 + 2 ∗ K[2] ∗ K[6] ∗ K[8] ∗ K[9]
− K[3] ∗ K[6] ∗ K[7] ∗ K[9] + 2 ∗ K[6]^3 ∗ K[7] + 2 ∗ U[2] ∗ X[2] ∗ K[3]
+ 2 ∗ U[2] ∗ K[8] ∗ K[9] − 2 ∗ X[1] ∗ K[6] ∗ K[9] − 4 ∗ X[2] ∗ K[6] ∗ K[8] :

SY[46] := 2 ∗ V[3] ∗ W[1] − U[2] ∗ K[2]^2 ∗ K[3] ∗ K[6] + K[2]^2 ∗ K[3] ∗ K[8] ∗ K[9]
− 2 ∗ K[3]^2 ∗ KK[5]^2 ∗ K[9] + 2 ∗ K[3] ∗ KK[5]^2 ∗ K[6]^2 + U[1] ∗ U[2] ∗ K[3] ∗ KK[5]
+ 2 ∗ U[2] ∗ K[2] ∗ K[6] ∗ K[8] + U[2] ∗ K[3] ∗ K[6] ∗ K[7] − U[1] ∗ KK[5] ∗ K[6] ∗ K[8]
− 2 ∗ K[2] ∗ K[8]^2 ∗ K[9] − K[3] ∗ K[7] ∗ K[8] ∗ K[9] − 2 ∗ U[2] ∗ X[1] ∗ K[6]
+ 2 ∗ X[1] ∗ K[8] ∗ K[9] :

SY[47] := 2 ∗ V[4] ∗ W[1] + U[2] ∗ K[2]^2 ∗ K[3] ∗ K[8] − 2 ∗ U[2] ∗ K[3]^2 ∗ KK[5]^2
− X[1] ∗ K[2]^2 ∗ K[3] ∗ K[6] + 2 ∗ K[3] ∗ KK[5]^2 ∗ K[6] ∗ K[8] − 2 ∗ U[2] ∗ K[2] ∗ K[8]^2
− U[2] ∗ K[3] ∗ K[7] ∗ K[8] + U[1] ∗ X[1] ∗ K[3] ∗ KK[5] − U[1] ∗ KK[5] ∗ K[8]^2
+ 2 ∗ X[1] ∗ K[2] ∗ K[6] ∗ K[8] + X[1] ∗ K[3] ∗ K[6] ∗ K[7] + 2 ∗ U[2] ∗ X[1] ∗ K[8]
− 2 ∗ X[1]^2 ∗ K[6] :

SY[48] := 2 ∗ V[1]^2 + K[2]^2 ∗ K[4] ∗ K[6]^2 + 2 ∗ K[3] ∗ K[4]^2 ∗ KK[5]^2
− 2 ∗ U[1] ∗ K[4] ∗ KK[5] ∗ K[6] − 2 ∗ K[2] ∗ K[4] ∗ K[8] ∗ K[9] − K[4] ∗ K[6]^2 ∗ K[7]
− 4 ∗ U[2] ∗ K[6] ∗ K[9] + 2 ∗ X[1] ∗ K[4] ∗ K[9] + 2 ∗ X[2] ∗ K[6]^2 + 2 ∗ K[8] ∗ K[9]^2 :

SY[49] := 2 ∗ V[1] ∗ V[2] + K[2]^2 ∗ K[4] ∗ K[6] ∗ K[8] − 2 ∗ K[2]^2 ∗ K[6]^3
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− 2 ∗ K[3] ∗ K[4] ∗ KK[5]^2 ∗ K[6] − 2 ∗ U[2] ∗ K[2] ∗ K[4] ∗ K[8] + 2 ∗ U[2] ∗ K[2] ∗ K[6]^2
− U[1] ∗ K[4] ∗ KK[5] ∗ K[8] + 3 ∗ U[1] ∗ KK[5] ∗ K[6]^2 + 2 ∗ K[2] ∗ K[6] ∗ K[8] ∗ K[9]
− K[4] ∗ K[6] ∗ K[7] ∗ K[8] + 2 ∗ K[6]^3 ∗ K[7] + 2 ∗ U[2] ∗ X[1] ∗ K[4] + 2 ∗ U[2] ∗ K[8] ∗ K[9]
− 4 ∗ X[1] ∗ K[6] ∗ K[9] − 2 ∗ X[2] ∗ K[6] ∗ K[8] :
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SY[50] := 2 ∗ V[1] ∗ W[3] − U[2] ∗ K[2]^2 ∗ K[4] ∗ K[6] + K[2]^2 ∗ K[4] ∗ K[8] ∗ K[9]
− 2 ∗ K[4]^2 ∗ KK[5]^2 ∗ K[8] + 2 ∗ K[4] ∗ KK[5]^2 ∗ K[6]^2 + U[1] ∗ U[2] ∗ K[4] ∗ KK[5]
+ 2 ∗ U[2] ∗ K[2] ∗ K[6] ∗ K[9] + U[2] ∗ K[4] ∗ K[6] ∗ K[7] − U[1] ∗ KK[5] ∗ K[6] ∗ K[9]
− 2 ∗ K[2] ∗ K[8] ∗ K[9]^2 − K[4] ∗ K[7] ∗ K[8] ∗ K[9] − 2 ∗ U[2] ∗ X[2] ∗ K[6]
+ 2 ∗ X[2] ∗ K[8] ∗ K[9] :
```

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SY[51] := 2 ∗ V[1] ∗ W[4] + U[2] ∗ K[2]^2 ∗ K[4] ∗ K[9] − 2 ∗ U[2] ∗ K[4]^2 ∗ KK[5]^2
− X[2] ∗ K[2]^2 ∗ K[4] ∗ K[6] + 2 ∗ K[4] ∗ KK[5]^2 ∗ K[6] ∗ K[9] − 2 ∗ U[2] ∗ K[2] ∗ K[9]^2
− U[2] ∗ K[4] ∗ K[7] ∗ K[9] + U[1] ∗ X[2] ∗ K[4] ∗ KK[5] − U[1] ∗ KK[5] ∗ K[9]^2
+ 2 ∗ X[2] ∗ K[2] ∗ K[6] ∗ K[9] + X[2] ∗ K[4] ∗ K[6] ∗ K[7] + 2 ∗ U[2] ∗ X[2] ∗ K[9]
− 2 ∗ X[2]^2 ∗ K[6] :
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SY[52] := 2 ∗ V[2]^2 + K[2]^2 ∗ K[4] ∗ K[8]^2 + 2 ∗ K[3] ∗ KK[5]^2 ∗ K[6]^2
− 2 ∗ U[1] ∗ KK[5] ∗ K[6] ∗ K[8] − 2 ∗ X[1] ∗ K[2] ∗ K[4] ∗ K[8] + 2 ∗ X[1] ∗ K[2] ∗ K[6]^2
− 2 ∗ K[2] ∗ K[8]^2 ∗ K[9] − K[4] ∗ K[7] ∗ K[8]^2 − 4 ∗ U[2] ∗ X[1] ∗ K[6] + 2 ∗ X[1]^2 ∗ K[4]
+ 2 ∗ X[1] ∗ K[8] ∗ K[9] + 2 ∗ X[2] ∗ K[8]^2 :
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SY[53] := 2 ∗ V[2] ∗ W[3] − K[2]^3 ∗ K[6]^3 − 2 ∗ K[2] ∗ K[3] ∗ K[4] ∗ KK[5]^2 ∗ K[6]
+ U[2] ∗ K[2]^2 ∗ K[4] ∗ K[8] + 2 ∗ U[2] ∗ K[2]^2 ∗ K[6]^2 + 2 ∗ U[1] ∗ K[2] ∗ KK[5] ∗ K[6]^2
− X[1] ∗ K[2]^2 ∗ K[4] ∗ K[6] + K[2]^2 ∗ K[6] ∗ K[8] ∗ K[9] + K[2] ∗ K[6]^3 ∗ K[7]
+ 2 ∗ K[3] ∗ KK[5]^2 ∗ K[6] ∗ K[9] − 3 ∗ U[1] ∗ U[2] ∗ KK[5] ∗ K[6]
− 2 ∗ U[2] ∗ K[2] ∗ K[8] ∗ K[9] − U[2] ∗ K[4] ∗ K[7] ∗ K[8] − 2 ∗ U[2] ∗ K[6]^2 ∗ K[7]
+ U[1] ∗ X[1] ∗ K[4] ∗ KK[5] + X[1] ∗ K[4] ∗ K[6] ∗ K[7] − 2 ∗ X[2] ∗ K[2] ∗ K[6] ∗ K[8]
+ K[6] ∗ K[7] ∗ K[8] ∗ K[9] + 2 ∗ U[2] ∗ X[2] ∗ K[8] :
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SY[54] := 2 ∗ V[2] ∗ W[4] + K[2]^4 ∗ K[4] ∗ K[6]^2 + 2 ∗ K[2]^2 ∗ K[3] ∗ K[4]^2 ∗ KK[5]^2
− U[2] ∗ K[2]^3 ∗ K[4] ∗ K[6] − 2 ∗ U[1] ∗ K[2]^2 ∗ K[4] ∗ KK[5] ∗ K[6]
− K[2]^3 ∗ K[4] ∗ K[8] ∗ K[9] − K[2]^3 ∗ K[6]^2 ∗ K[9] − 2 ∗ K[2]^2 ∗ K[4] ∗ K[6]^2 ∗ K[7]
− 2 ∗ K[2] ∗ K[3] ∗ K[4] ∗ KK[5]^2 ∗ K[9] − 2 ∗ K[2] ∗ K[4]^2 ∗ KK[5]^2 ∗ K[8]
+ 2 ∗ K[2] ∗ K[4] ∗ KK[5]^2 ∗ K[6]^2 − 2 ∗ K[3] ∗ K[4]^2 ∗ KK[5]^2 ∗ K[7]
+ U[1] ∗ U[2] ∗ K[2] ∗ K[4] ∗ KK[5] + U[2] ∗ K[2] ∗ K[4] ∗ K[6] ∗ K[7]
− 6 ∗ U[2] ∗ K[4] ∗ KK[5]^2 ∗ K[6] + U[1] ∗ K[2] ∗ KK[5] ∗ K[6] ∗ K[9]
+2 ∗ U[1] ∗ K[4] ∗ KK[5] ∗ K[6] ∗ K[7]+X[1] ∗ K[2]^2 ∗ K[4] ∗ K[9]+2 ∗ X[1] ∗ K[4]^2 ∗ KK[5]^2
+ X[2] ∗ K[2]^2 ∗ K[4] ∗ K[8] + 2 ∗ X[2] ∗ K[2]^2 ∗ K[6]^2 + 4 ∗ X[2] ∗ K[3] ∗ K[4] ∗ KK[5]^2
+ K[2]^2 ∗ K[8] ∗ K[9]^2 + 2 ∗ X[2]^2 ∗ K[8] + K[2] ∗ K[4] ∗ K[7] ∗ K[8] ∗ K[9]
+ K[2] ∗ K[6]^2 ∗ K[7] ∗ K[9] − 2 ∗ K[3] ∗ KK[5]^2 ∗ K[9]^2 + K[4] ∗ K[6]^2 ∗ K[7]^2
+ 4 ∗ KK[5]^2 ∗ K[6]^2 ∗ K[9] + U[1] ∗ U[2] ∗ KK[5] ∗ K[9] − 2 ∗ U[2] ∗ X[2] ∗ K[2] ∗ K[6]
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+ 2 * U[2] * K[6] * K[7] * K[9] − 3 * U[1] * X[2] * KK[5] * K[6] − X[1] * K[4] * K[7] * K[9]
− 2 * X[2] * K[2] * K[8] * K[9] − X[2] * K[4] * K[7] * K[8] − 2 * X[2] * K[6]^2 * K[7]
− K[7] * K[8] * K[9]^2 :

SY[55] := 2 * W[2]^2 + K[2]^2 * K[3] * K[9]^2 + 2 * K[4] * KK[5]^2 * K[6]^2
− 2 * U[1] * KK[5] * K[6] * K[9] − 2 * X[2] * K[2] * K[3] * K[9] + 2 * X[2] * K[2] * K[6]^2
− 2 * K[2] * K[8] * K[9]^2 − K[3] * K[7] * K[9]^2 − 4 * U[2] * X[2] * K[6] + 2 * X[1] * K[9]^2
+ 2 * X[2]^2 * K[3] + 2 * X[2] * K[8] * K[9] :

SY[56] := 2 * V[3] * W[2] − K[2]^3 * K[6]^3 − 2 * K[2] * K[3] * K[4] * KK[5]^2 * K[6]
+ U[2] * K[2]^2 * K[3] * K[9] + 2 * U[2] * K[2]^2 * K[6]^2 + 2 * U[1] * K[2] * KK[5] * K[6]^2
− X[2] * K[2]^2 * K[3] * K[6] + K[2]^2 * K[6] * K[8] * K[9] + K[2] * K[6]^3 * K[7]
+ 2 * K[4] * KK[5]^2 * K[6] * K[8] − 3 * U[1] * U[2] * KK[5] * K[6]
− 2 * U[2] * K[2] * K[8] * K[9] − U[2] * K[3] * K[7] * K[9] − 2 * U[2] * K[6]^2 * K[7]
+ U[1] * X[2] * K[3] * KK[5] − 2 * X[1] * K[2] * K[6] * K[9] + X[2] * K[3] * K[6] * K[7]
+ K[6] * K[7] * K[8] * K[9] + 2 * U[2] * X[1] * K[9] :

SY[57] := 2 * V[4] * W[2] + K[2]^4 * K[3] * K[6]^2 + 2 * K[2]^2 * K[3]^2 * K[4] * KK[5]^2
− U[2] * K[2]^3 * K[3] * K[6] − 2 * U[1] * K[2]^2 * K[3] * KK[5] * K[6]
− K[2]^3 * K[3] * K[8] * K[9] − K[2]^3 * K[6]^2 * K[8] − 2 * K[2]^2 * K[3] * K[6]^2 * K[7]
− 2 * K[2] * K[3]^2 * KK[5]^2 * K[9] − 2 * K[2] * K[3] * K[4] * KK[5]^2 * K[8]
+ 2 * K[2] * K[3] * KK[5]^2 * K[6]^2 − 2 * K[3]^2 * K[4] * KK[5]^2 * K[7]
+ U[1] * U[2] * K[2] * K[3] * KK[5] + U[2] * K[2] * K[3] * K[6] * K[7]
− 6 * U[2] * K[3] * KK[5]^2 * K[6] + U[1] * K[2] * KK[5] * K[6] * K[8]
+2 * U[1] * K[3] * KK[5] * K[6] * K[7]+X[1] * K[2]^2 * K[3] * K[9]+2 * X[1] * K[2]^2 * K[6]^2
+ 4 * X[1] * K[3] * K[4] * KK[5]^2 + X[2] * K[2]^2 * K[3] * K[8] + 2 * X[2] * K[3]^2 * KK[5]^2
+ K[2]^2 * K[8]^2 * K[9] + K[2] * K[3] * K[7] * K[8] * K[9] + K[2] * K[6]^2 * K[7] * K[8]
+ K[3] * K[6]^2 * K[7]^2 − 2 * K[4] * KK[5]^2 * K[8]^2 + 4 * KK[5]^2 * K[6]^2 * K[8]
+ U[1] * U[2] * KK[5] * K[8] − 2 * U[2] * X[1] * K[2] * K[6] + 2 * U[2] * K[6] * K[7] * K[8]
− 3 * U[1] * X[1] * KK[5] * K[6] − 2 * X[1] * K[2] * K[8] * K[9] − X[1] * K[3] * K[7] * K[9]
−2 * X[1] * K[6]^2 * K[7]−X[2] * K[3] * K[7] * K[8]−K[7] * K[8]^2 * K[9] + 2 * X[1]^2 * K[9]:

SY[58] := 4 * V[3]^2 + K[2]^4 * K[3] * K[6]^2 + 2 * K[2]^2 * K[3]^2 * K[4] * KK[5]^2
− 2 * U[1] * K[2]^2 * K[3] * KK[5] * K[6] − 2 * K[2]^3 * K[3] * K[8] * K[9]
− 2 * K[2]^3 * K[6]^2 * K[8] − 2 * K[2]^2 * K[3] * K[6]^2 * K[7]
− 4 * K[2] * K[3] * K[4] * KK[5]^2 * K[8] − 2 * K[3]^2 * K[4] * KK[5]^2 * K[7]
− 8 * U[2] * K[3] * KK[5]^2 * K[6] + 4 * U[1] * K[2] * KK[5] * K[6] * K[8]
+ 2 * U[1] * K[3] * KK[5] * K[6] * K[7] + 2 * X[1] * K[2]^2 * K[3] * K[9]
+ 2 * X[1] * K[2]^2 * K[6]^2 + 4 * X[1] * K[3] * K[4] * KK[5]^2 + 4 * X[2] * K[3]^2 * KK[5]^2
+ 4 * K[2]^2 * K[8]^2 * K[9] + 2 * K[2] * K[3] * K[7] * K[8] * K[9]

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+ 2 * K[2] * K[6]^2 * K[7] * K[8] + K[3] * K[6]^2 * K[7]^2 + 4 * KK[5]^2 * K[6]^2 * K[8]
− 4 * U[1] * X[1] * KK[5] * K[6] − 8 * X[1] * K[2] * K[8] * K[9]
− 2 * X[1] * K[3] * K[7] * K[9] − 2 * X[1] * K[6]^2 * K[7] + 4 * X[1]^2 * K[9] :

SY[59] := 4 * V[3] * V[4] + K[2]^4 * K[3] * K[6] * K[8]−2 * K[2]^2 * K[3]^2 * KK[5]^2 * K[6]
− 2 * U[2] * K[2]^3 * K[3] * K[8] + 4 * U[2] * K[2] * K[3]^2 * KK[5]^2
− U[1] * K[2]^2 * K[3] * KK[5] * K[8] + 2 * U[1] * K[3]^2 * KK[5]^3
− 2 * K[2]^3 * K[6] * K[8]^2 − 2 * K[2]^2 * K[3] * K[6] * K[7] * K[8]
+ 2 * K[3]^2 * KK[5]^2 * K[6] * K[7] + 2 * U[2] * X[1] * K[2]^2 * K[3]
+ 4 * U[2] * K[2]^2 * K[8]^2 + 2 * U[2] * K[2] * K[3] * K[7] * K[8]
− 4 * U[2] * K[3] * KK[5]^2 * K[8] + 2 * U[1] * K[2] * KK[5] * K[8]^2
+ U[1] * K[3] * KK[5] * K[7] * K[8] + 2 * X[1] * K[2]^2 * K[6] * K[8]
− 4 * X[1] * K[3] * KK[5]^2 * K[6] + 2 * K[2] * K[6] * K[7] * K[8]^2
+ K[3] * K[6] * K[7]^2 * K[8] + 4 * KK[5]^2 * K[6] * K[8]^2 − 8 * U[2] * X[1] * K[2] * K[8]
− 2 * U[2] * X[1] * K[3] * K[7] − 2 * U[1] * X[1] * KK[5] * K[8]
− 2 * X[1] * K[6] * K[7] * K[8] + 4 * U[2] * X[1]^2 :

SY[60] := 4 * W[3]^2 + K[2]^4 * K[4] * K[6]^2 + 2 * K[2]^2 * K[3] * K[4]^2 * KK[5]^2
− 2 * U[1] * K[2]^2 * K[4] * KK[5] * K[6] − 2 * K[2]^3 * K[4] * K[8] * K[9]
− 2 * K[2]^3 * K[6]^2 * K[9] − 2 * K[2]^2 * K[4] * K[6]^2 * K[7]
− 4 * K[2] * K[3] * K[4] * KK[5]^2 * K[9] − 2 * K[3] * K[4]^2 * KK[5]^2 * K[7]
− 8 * U[2] * K[4] * KK[5]^2 * K[6] + 4 * U[1] * K[2] * KK[5] * K[6] * K[9]
+ 2 * U[1] * K[4] * KK[5] * K[6] * K[7] + 4 * X[1] * K[4]^2 * KK[5]^2
+ 2 * X[2] * K[2]^2 * K[4] * K[8] + 2 * X[2] * K[2]^2 * K[6]^2 + 4 * X[2]^2 * K[8]
+ 4 * X[2] * K[3] * K[4] * KK[5]^2 + 4 * K[2]^2 * K[8] * K[9]^2
+ 2 * K[2] * K[4] * K[7] * K[8] * K[9] + 2 * K[2] * K[6]^2 * K[7] * K[9]
+ K[4] * K[6]^2 * K[7]^2 + 4 * KK[5]^2 * K[6]^2 * K[9] − 4 * U[1] * X[2] * KK[5] * K[6]
− 8 * X[2] * K[2] * K[8] * K[9] − 2 * X[2] * K[4] * K[7] * K[8] − 2 * X[2] * K[6]^2 * K[7] :

SY[61] := 4 * W[3] * W[4] + K[2]^4 * K[4] * K[6] * K[9]−2 * K[2]^2 * K[4]^2 * KK[5]^2 * K[6]
− 2 * U[2] * K[2]^3 * K[4] * K[9] + 4 * U[2] * K[2] * K[4]^2 * KK[5]^2
− U[1] * K[2]^2 * K[4] * KK[5] * K[9] + 2 * U[1] * K[4]^2 * KK[5]^3
− 2 * K[2]^3 * K[6] * K[9]^2 − 2 * K[2]^2 * K[4] * K[6] * K[7] * K[9]
+ 2 * K[4]^2 * KK[5]^2 * K[6] * K[7] + 2 * U[2] * X[2] * K[2]^2 * K[4]
+ 4 * U[2] * K[2]^2 * K[9]^2 + 2 * U[2] * K[2] * K[4] * K[7] * K[9]
− 4 * U[2] * K[4] * KK[5]^2 * K[9] + 2 * U[1] * K[2] * KK[5] * K[9]^2
+ U[1] * K[4] * KK[5] * K[7] * K[9] + 2 * X[2] * K[2]^2 * K[6] * K[9]
− 4 * X[2] * K[4] * KK[5]^2 * K[6] + 2 * K[2] * K[6] * K[7] * K[9]^2
+ K[4] * K[6] * K[7]^2 * K[9] + 4 * KK[5]^2 * K[6] * K[9]^2 − 8 * U[2] * X[2] * K[2] * K[9]
− 2 * U[2] * X[2] * K[4] * K[7] − 2 * U[1] * X[2] * KK[5] * K[9]
− 2 * X[2] * K[6] * K[7] * K[9] + 4 * U[2] * X[2]^2 :
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SY[62] := 4 * V[4]^2 + K[2]^4 * K[3] * K[8]^2 − 4 * K[2]^2 * K[3]^2 * KK[5]^2 * K[8]
+ 4 * K[3]^3 * KK[5]^4 − 2 * X[1] * K[2]^3 * K[3] * K[8] + 4 * X[1] * K[2] * K[3]^2 * KK[5]^2
− 2 * K[2]^3 * K[8]^3 − 2 * K[2]^2 * K[3] * K[7] * K[8]^2 + 4 * K[2] * K[3] * KK[5]^2 * K[8]^2
+ 4 * K[3]^2 * KK[5]^2 * K[7] * K[8] + 2 * X[1]^2 * K[2]^2 * K[3] + 6 * X[1] * K[2]^2 * K[8]^2
+ 2 * X[1] * K[2] * K[3] * K[7] * K[8] − 12 * X[1] * K[3] * KK[5]^2 * K[8]
+ 2 * K[2] * K[7] * K[8]^3 + K[3] * K[7]^2 * K[8]^2 + 4 * KK[5]^2 * K[8]^3
− 8 * X[1]^2 * K[2] * K[8] − 2 * X[1]^2 * K[3] * K[7] − 2 * X[1] * K[7] * K[8]^2 + 4 * X[1]^3 :

SY[63] := 4 * W[4]^2 + K[2]^4 * K[4] * K[9]^2 − 4 * K[2]^2 * K[4]^2 * KK[5]^2 * K[9]
+ 4 * K[4]^3 * KK[5]^4 − 2 * X[2] * K[2]^3 * K[4] * K[9] + 4 * X[2] * K[2] * K[4]^2 * KK[5]^2
− 2 * K[2]^3 * K[9]^3 − 2 * K[2]^2 * K[4] * K[7] * K[9]^2 + 4 * K[2] * K[4] * KK[5]^2 * K[9]^2
+ 4 * K[4]^2 * KK[5]^2 * K[7] * K[9] + 2 * X[2]^2 * K[2]^2 * K[4] + 6 * X[2] * K[2]^2 * K[9]^2
+ 2 * X[2] * K[2] * K[4] * K[7] * K[9] − 12 * X[2] * K[4] * KK[5]^2 * K[9]
+ 2 * K[2] * K[7] * K[9]^3 + K[4] * K[7]^2 * K[9]^2 + 4 * KK[5]^2 * K[9]^3
− 8 * X[2]^2 * K[2] * K[9] − 2 * X[2]^2 * K[4] * K[7] − 2 * X[2] * K[7] * K[9]^2 + 4 * X[2]^3 :
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